

# A Polynomial Invariant for Rank 3 Weakly-Colored Stranded Graphs

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**ABSTRACT.** The Bollobas-Riordan polynomial [Math. Ann. 323, 81 (2002)] is a universal polynomial invariant for ribbon graphs. We find an extension of this polynomial for a particular family of graphs called rank 3 weakly-colored stranded graphs. These graphs live in a 3D space and appear as the gluing stranded vertices with stranded edges according to a definite rule (ordinary graphs and ribbon graphs can be understood in terms of stranded graphs as well). They also possess a color structure in a specific sense [Gurau, Commun. Math. Phys. 304, 69 (2011)]. The polynomial constructed is a seven indeterminate polynomial invariant of these graphs which responds to a similar contraction/deletion recurrence relation as obeyed by the Tutte and Bollobas-Riordan polynomials. It is however new due to the particular cellular structure of the graphs on which it relies. The present polynomial encodes therefore additional data that neither the Tutte nor the Bollobas-Riordan polynomials can capture for the type of graphs described in the present work.

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## 1. Background and main results

The Tutte polynomial is a universal polynomial invariant defined on a graph which obeys a particular recurrence relation for the contraction and deletion of the edges of this graph. This universal property turns out to have several implications in particular in statistical mechanics [17]. The Bollobas-Riordan (BR) polynomial [6] defines an authentic extension of the Tutte polynomial for simple graphs [19] to ribbon graphs or neighborhoods of graphs embedded in surfaces.

In this work, we investigate a family of graphs for which both the Tutte and BR polynomial invariants find a meaningful extension. The family of such graphs are issued from the so-called colored tensor graphs representing simplicial complexes in any dimension  $D$  [10]. The present study is mainly addressed for  $D = 3$ , although, at the beginning and for the sake of generality, we carry out the analysis for any dimension  $D$ . We focus on  $D = 3$  because we aim at making a first step towards the definition of a universal polynomial invariant precisely of the Tutte or BR form for particular graphs having a (3D) volume which is still missing to the best of our knowledge.

It can be underlined that the graphs that we shall study are somehow recently found. They naturally appear as Feynman graphs in a particular quantum field theoretical framework called colored tensor field theory introduced by Gurau [10][12] and further investigated afterwards [11, 3, 2, 13]. For a version of these graphs without color, we refer the reader to more references therein. According to quantum field procedures, these graphs can be dually associated with gluing of basic  $D$  simplexes (vertices) along their faces or  $D - 1$  boundary simplexes (propagators or edges). The graphs in question are called stranded and can be depicted in a three dimensional space. Fields in this framework are rank  $D$  tensors. Hence the notion of (tensor) rank and the dimension of the space generated by the gluing are related. We shall not distinguish these in the following.

There are two main contributions addressing an extension of the BR polynomial for a similar family of graphs that will be presented here: the contribution by Gurau [12] and the one by Tanasa [18]. Let us recall that for tensor graphs, the vertex is stranded and the simplest way of contracting an edge leads immediately to another type of vertex which differs from the initial. As a consequence, the simplest prescriptions of contracting and deleting edges are not well defined without any further considerations. And indeed, the two aforementioned works undertake the definition of the contraction and deletion of edges in a very peculiar manner.

There are several layers of difficulties in order to obtain the correct definition of the type of graphs that ought to be considered, the type of topological polynomial defined for this family of graphs and the type of recurrence relations that the polynomial needs to satisfy. Our method is radically different from that of Tanasa since we will use the Gurau colored prescription in order to have a control on the type of graphs that could be generated. Moreover, and in contrast with the work by Gurau, we propose to enlarge the family of graphs from the colored tensor graphs to what will be called weakly-colored (w-colored) stranded graphs for which contraction and a similar notion of deletion make a sense. Strictly speaking these w-colored stranded graphs are equivalence classes of graphs. Within these classes, the contraction operation becomes stable in some sense. From this stability, we can define a polynomial containing all specific data pertaining to the extended dimension 3 and the associated cellular decomposition of the graph. We can establish as well a recurrence relation satisfied by the new polynomial for w-colored stranded graphs.

Our main result appears in Theorem 4 of Section 4. This statement successfully determines the contraction and cut rule (an operation intuitively clear which replaces the deletion) with respect to an edge that should be fulfilled by a new polynomial spelled out in Definition 36. The representatives of the equivalence class of graphs on which the polynomial is defined have several topological ingredients. Some of these ingredients will be introduced step by step in the following program as they turn out to be well defined for simple graphs and ribbon graphs.

A first ingredient is captured by the notion of flag [19] which can be introduced at the simplest graph level. This analysis is the purpose of Section 2. Theorem 1 yields another interesting result of this work since it mainly establishes that the Tutte polynomial for graphs with flags satisfies the universal equations of the Tutte polynomial. This point is original, to our knowledge, since it proves that the universal equations for the Tutte polynomial with particular weights, apart from their more general meaning and usefulness in statistical mechanics [5], can be found as equations for simple graphs decorated with flags.

A second key notion is that of open and closed faces of a graph which can be already introduced for ribbon graphs. The consequences that open and closed faces in a ribbon graph have on the related BR polynomial are addressed in Section 3. Theorem 3 is another partial important contributing in establishing our final result. We emphasize that in the work by Krajewski and co-workers [15] ribbon graphs with flags and their topological polynomial have been introduced in a different manner (the notion of open faces is definitely absent therein, for instance). These authors used the framework of the partial or Chmutov duality [9] for establishing several interesting results pertaining to ribbon graphs with flags. Our approach is more “conventional” and much in the spirit of the contribution [6]. Thus, as far as we are concerned with ribbon graph flags and to the

extent of this work, Theorem 3 is new and explicitly describes the implications that might have flags on the BR polynomial.

A closing appendix provides explicit examples of the polynomial obtained for typical graphs.

## 2. Graphs and Tutte polynomial

This section starts the review of the main theorem that we aim to generalize in the next sections.

**2.1. Graphs.** We review basic definitions for graphs (called sometimes simple graphs in the following) and their Tutte polynomial [19, 5, 16].

**DEFINITION 1** (Graph [19]). *A graph  $\mathcal{G}$  is defined by a set  $\mathcal{V}$  of vertices and a set  $\mathcal{E}$  of edges and a relation of incidence which associates with each edge either one or two vertices called its ends. We denote it as  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  or simply  $\mathcal{G}$ .*

**DEFINITION 2** (Edges and terminal forms [19][16]). *Let  $e$  be an edge of a graph  $\mathcal{G}$ .*

- *$e$  is called a self-loop in  $\mathcal{G}$  if the two ends of  $e$  are adjacent to the same vertex  $v$  of  $\mathcal{G}$ .*
- *$e$  is called a bridge in  $\mathcal{G}$  if its removal disconnects a component of  $\mathcal{G}$ .*
- *$e$  is called an ordinary or regular edge of  $\mathcal{G}$  if it is neither a bridge nor a self-loop.*
- *A graph which does not contain any regular edge is called a terminal form.*

**DEFINITION 3** (Deletion and contraction [19], Chap. I and II). *Let  $\mathcal{G}$  be a graph and  $e$  one of its edges.*

- *We call  $\mathcal{G} - e$  the graph obtained from  $\mathcal{G}$  by removing  $e$ .*
- *If  $e$  is not a self-loop, the graph  $\mathcal{G}/e$  obtained by contracting  $e$  is defined from  $\mathcal{G}$  by deleting  $e$  and identifying its end vertices into a new vertex.*
- *If  $e$  is a self-loop,  $\mathcal{G}/e$  is by definition the same as  $\mathcal{G} - e$ .*

One notices that after an arbitrary full contraction-deletion of regular edges of a given graph the end result is necessarily given by a collection of graphs composed by bridges and/or self-loops, hence a terminal form.

**DEFINITION 4** (Subgraphs, spanning subgraphs [19]). • *A subgraph  $A$  of  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  is defined as a graph  $A(\mathcal{V}_A, \mathcal{E}_A)$  the vertex set of which is a subset of  $\mathcal{V}$  and the edge set of which is a subset of  $\mathcal{E}$  together with their end vertices. Hence,  $\mathcal{E}_A \subseteq \mathcal{E}$  and  $\mathcal{V}_A \subseteq \mathcal{V}$  and we denote  $A \subseteq \mathcal{G}$ .*

- *A spanning subgraph  $A$  of  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  is defined as a subgraph  $A(\mathcal{V}_A, \mathcal{E}_A)$  of  $\mathcal{G}$  with all vertices of  $\mathcal{G}$ . Hence  $\mathcal{E}_A \subseteq \mathcal{E}$  and  $\mathcal{V}_A = \mathcal{V}$  and we denote  $A \in \mathcal{G}$ .*

**DEFINITION 5** (Graph operations [19][6]). *Consider two disjoint graphs  $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$  and  $\mathcal{G}_2(\mathcal{V}_2, \mathcal{E}_2)$ . We define*

- *The disjoint union graph  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  as the graph defined by the disjoint union of the two graphs;*
- *The product graph  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  of the two graphs simply merges at the vertices  $v_1 \in \mathcal{V}_1$  and  $v_2 \in \mathcal{V}_2$  the two graphs by removing  $v_1$  and  $v_2$  and insert a new vertex  $v$  which has all edges incident to  $v_1$  and to  $v_2$ .*

Note that if a graph  $\mathcal{G}$  can be written as  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$ , this also means that  $\mathcal{G}$  is one-to-one with  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  (we will come back on this bijection later). It is simple to check that  $A \in \mathcal{G}_1 \sqcup \mathcal{G}_2$  can be written as  $A_1 \sqcup A_2$ , where  $A_i \in \mathcal{G}_i$ . A similar relation holds for subgraphs  $A \subset \mathcal{G}_1 \sqcup \mathcal{G}_2$ . Idem,  $A \in \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  implies that  $A = A_1 \cdot_{v_1, v_2} A_2$  where  $A_i \in \mathcal{G}_i$ . However such a relation holds for subgraphs of  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  only if each  $A_i \subset \mathcal{G}_i$  contains the vertex  $v_i$ .

We are now in position to define the Tutte polynomial and specify its main properties.

**DEFINITION 6** (Tutte polynomial 1 [6][16]). *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  be a graph, then the Tutte polynomial  $T_{\mathcal{G}}$  of  $\mathcal{G}$  admits the following expansion:*

$$T_{\mathcal{G}}(x, y) = \sum_{A \in \mathcal{G}} (x-1)^{r(\mathcal{G})-r(A)} (y-1)^{n(A)}, \quad (1)$$

where  $r(A) = |\mathcal{V}| - k(A)$  is the rank of the spanning subgraph  $A$ ,  $k(A)$  its number of connected component and  $n(A) = |A| + k(A) - |\mathcal{V}|$  is its nullity or cyclomatic number.

There is an equivalent definition of this polynomial.

DEFINITION 7 (Tutte polynomial 2 [5]). *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  be a graph. The Tutte polynomial of  $\mathcal{G}$  denoted  $T_{\mathcal{G}}(x, y)$  is defined in the following way:*

- *If  $\mathcal{G}$  has no edges, then  $T_{\mathcal{G}}(x, y) = 1$ .*
- Otherwise, for any edge  $e \in \mathcal{E}$* 
  - *$T_{\mathcal{G}}(x, y) = T_{\mathcal{G}-e}(x, y) + T_{\mathcal{G}/e}(x, y)$ , if  $e$  is a regular edge.*
  - *$T_{\mathcal{G}}(x, y) = xT_{\mathcal{G}-e}(x, y) = xT_{\mathcal{G}/e}$ , if  $e$  is a bridge.*
  - *$T_{\mathcal{G}}(x, y) = yT_{\mathcal{G}-e}(x, y) = yT_{\mathcal{G}/e}(x, y)$ , if  $e$  is a self-loop.*

Some comments are in order at this point.

(1) In several works, the variable  $(y - 1)$  appearing in (1) is replaced simply by  $y^1$ . The above convention (for instance used in [16]) is simpler for the following reason. In order to compute the Tutte polynomial of  $\mathcal{G}$ , one performs a full contraction and deletion of regular edges of this graph which yields terminal forms. The Tutte polynomial of  $\mathcal{G}$  is obtained by summing the contributions of its terminal forms. According to the above convention,  $(x - 1)$  and  $(y - 1)$ , for a terminal form composed with  $m$  bridges and  $p$  self-loops, the Tutte polynomial is  $x^m y^p$ .

(2) By definition, for a self-loop  $e$ ,  $\mathcal{G} - e = \mathcal{G}/e$  hence  $T_{\mathcal{G}-e} = T_{\mathcal{G}/e}$  holds. It is however not trivial to prove that, for a bridge  $e$ ,  $T_{\mathcal{G}-e} = T_{\mathcal{G}/e}$  holds. A way to agree with this is to remark that  $\mathcal{G} - e = \mathcal{G}_1 \sqcup \mathcal{G}_2$  and  $\mathcal{G}/e = \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  where  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are the two maximal<sup>2</sup> subgraphs on each side of the bridge and  $v_1$  and  $v_2$  are the end vertices of  $e$ . The claim  $T_{\mathcal{G}-e} = T_{\mathcal{G}/e}$  is a direct consequence of the next proposition following from the additivity property of the rank and the nullity with respect to the disjoint union and product operations for graphs which implies a term by term monomial factorization.

PROPOSITION 1 (Operations on Tutte polynomial 1 [19]). *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two disjoint graphs, then*

$$T_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = T_{\mathcal{G}_1} T_{\mathcal{G}_2} = T_{\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2}. \quad (2)$$

*for any vertices  $v_{1,2}$  in  $\mathcal{G}_{1,2}$ , respectively.*

**2.2. Graphs with flags.** There exists a notion of “decorated” graph encompassing the above graph consideration. This new category includes graphs with flags or half-edges [16] hooked to vertices.

DEFINITION 8 (Flag). *A flag or half-edge (or half-line) is a line incident to a unique vertex and without forming a loop. (See Figure 1.)*



FIGURE 1. A flag incident to a unique vertex.

DEFINITION 9 (Cut of an edge [16]). *Let  $e \in \mathcal{E}$  be an edge of  $\mathcal{G}(\mathcal{V}, \mathcal{E})$ . The cut graph  $\mathcal{G} \vee e$  is the graph obtained from  $\mathcal{G}$  by “cutting”  $e$ , namely by removing  $e$  and let two flags attached to the end vertices of  $e$ . If  $e$  is a self-loop then the resulting two flags are attached to the same vertex. (See an illustration in Figure 2)*

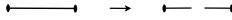


FIGURE 2. Cutting an edge.

DEFINITION 10 (Graph and subgraph with flags). • *A graph  $\mathcal{G}$  with flags is a graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  with a set  $\mathfrak{f}$  of flags defined by the disjoint union of  $\mathfrak{f}^1$  the set of flags obtained only from the cut of all edges of  $\mathcal{G}$  and a set  $\mathfrak{f}^0$  of “additional” flags together with a relation which associates with each*

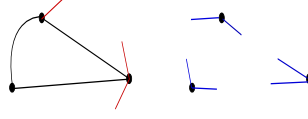


FIGURE 3. A graph with flags, its set  $f^0$  of additional flags in red and set  $f^1$  in blue after cutting its edges and removing  $f^0$ .

additional flag a unique vertex. We denote a graph with set  $f^0$  of additional flags as  $\mathcal{G}(\mathcal{V}, \mathcal{E}, f^0)$ . (See Figure 3.)

- A  $c$ -subgraph  $A$  of  $\mathcal{G}(\mathcal{V}, \mathcal{E}, f^0)$  is defined as a graph  $A(\mathcal{V}_A, \mathcal{E}_A, f_A^0)$  the vertex set of which is a subset of  $\mathcal{V}$ , the edge set of which is a subset of  $\mathcal{E}$  together with their end vertices. Call  $\mathcal{E}'_A$  the set of edges incident to the vertices of  $A$  and not contained in  $\mathcal{E}_A$ . The flag set of  $A$  contains a subset of  $f^0$  plus additional flags attached to the vertices of  $A$  obtained by cutting all edges in  $\mathcal{E}'_A$ . In symbols,  $\mathcal{E}_A \subseteq \mathcal{E}$  and  $\mathcal{V}_A \subseteq \mathcal{V}$ ,  $f_A^0 = f_A^{0;0} \cup f_A^{0;1}(\mathcal{E}_A)$  with  $f_A^{0;0} \subseteq f^0$  and  $f_A^{0;1}(\mathcal{E}_A) \subseteq f^1$ , where  $f_A^{0;1}(\mathcal{E}_A)$  is the set of flags obtained by cutting all edges in  $\mathcal{E}'_A$  and incident to vertices of  $A$ . We write  $A \subseteq \mathcal{G}$ . (See  $A$  as illustrated in Figure 4.)

- A spanning  $c$ -subgraph  $\tilde{A}$  of  $\mathcal{G}(\mathcal{V}, \mathcal{E}, f^0)$  is defined as a  $c$ -subgraph  $\tilde{A}(\mathcal{V}_A, \mathcal{E}_A, f_A^0)$  of  $\mathcal{G}$  with all vertices and all additional flags of  $\mathcal{G}$ . Hence  $\mathcal{E}_A \subseteq \mathcal{E}$  and  $\mathcal{V}_A = \mathcal{V}$ ,  $f_A^0 = f^0 \cup f_A^{0;1}(\mathcal{E}_A)$ . We denote it  $\tilde{A} \in \mathcal{G}$ . (See  $\tilde{A}$  as an illustration in Figure 4.)

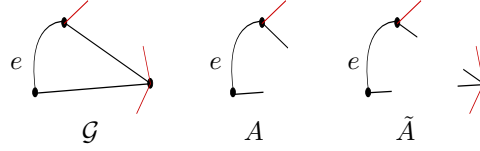


FIGURE 4. A graph  $\mathcal{G}$ , a  $c$ -subgraph  $A$  defined by the edge  $e$  and the spanning  $c$ -subgraph  $\tilde{A}$  associated with  $e$ .

Remark that a spanning  $c$ -subgraph has always a greater number of additional flags than the initial graph which spans it. Moreover, one can define the rank and nullity of any (spanning  $c$ -sub-) graph with flags as in the ordinary situation.

**DEFINITION 11** (Edge contraction of graphs with flags). Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, f^0)$  be a graph with flags. We define the contraction of a non self-loop edge  $e$  in  $\mathcal{G}$ , i.e.  $\mathcal{G}/e$ , by the graph obtained from  $\mathcal{G}$  by removing  $e$  and identifying the two end vertices into a new vertex having all their additional flags and remaining incident lines. For a self-loop  $e$ , contraction  $\mathcal{G}/e$  and deletion  $\mathcal{G} - e$  coincide.

Noting that the above contraction does not change the number of additional flags of a graph, the following proposition is straightforward.

**PROPOSITION 2.** Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, f^0)$  be a graph with flags and  $e$  be an edge. Then

$$f^0(\mathcal{G} \vee e) = f^0 \cup \{e_1, e_2\}, \quad f^0(\mathcal{G}/e) = f^0, \quad (3)$$

where the two flags  $e_{1,2}$  are obtained from cutting  $e$ .

Operations for graphs in the sense of Definition 5 can be reported for graphs with flags. The disjoint union graph  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  and product graph  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  keep the same sense with set of additional flags the disjoint union of additional flag sets. Spanning  $c$ -subgraphs of  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  and  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  are of the form  $A_1 \sqcup A_2$  and  $A_1 \cdot_{v_1, v_2} A_2$ , respectively, with  $A_i \in \mathcal{G}_i$ . The only point to check is that, given  $A_i(\mathcal{V}(\mathcal{G}_i), \mathcal{E}_{A_i}, f_{A_i}^0 = f_{\mathcal{G}_i}^0 \cup f_{A_i}^{0;1}(\mathcal{E}_{A_i}))$

$$f_{A_1 \sqcup A_2}^0 = f_{\mathcal{G}_1}^0 \cup f_{\mathcal{G}_2}^0, \quad f_{A_1 \sqcup A_2}^{0;1}(\mathcal{E}_{A_1} \cup \mathcal{E}_{A_2}) = f_{A_1}^{0;1}(\mathcal{E}_{A_1}) \cup f_{A_2}^{0;1}(\mathcal{E}_{A_2}) \quad (4)$$

$$f_{A_1 \cdot_{v_1, v_2} A_2}^0 = f_{\mathcal{G}_1}^0 \cup f_{\mathcal{G}_2}^0, \quad f_{A_1 \cdot_{v_1, v_2} A_2}^{0;1}(\mathcal{E}_{A_1} \cup \mathcal{E}_{A_2}) = f_{A_1}^{0;1}(\mathcal{E}_{A_1}) \cup f_{A_2}^{0;1}(\mathcal{E}_{A_2}). \quad (5)$$

<sup>1</sup>Using  $y$  instead of  $(y - 1)$  leads to non negative coefficients of the polynomial [6].

<sup>2</sup>These graphs are maximal in the sense that they contain all vertices and all lines on a given side of the bridge.

This holds because the flags obtained by cutting  $(\mathcal{E}_{A_1} \cup \mathcal{E}_{A_2})'$  in  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  or by cutting  $(\mathcal{E}_{A_1} \cup \mathcal{E}_{A_2})'$  in  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  turns out to be identical to the flags obtained from cutting  $\mathcal{E}'_{A_1}$  in  $\mathcal{G}_1$  (letting  $\mathcal{G}_2$  untouched) with those from cutting  $\mathcal{E}'_{A_2}$  in  $\mathcal{G}_2$  (letting  $\mathcal{G}_1$  untouched). One finally shows that spanning c-subgraphs of  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  and of  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  are one-to-one.

We can now identify a new polynomial for graphs with flags and list its main properties.

**DEFINITION 12** (Tutte polynomial for graphs with flags). *Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathbf{f}^0)$  be a graph with flags. The Tutte polynomial of  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathbf{f}^0)$  is given by*

$$\mathcal{T}_{\mathcal{G}}(x, y, t) = \sum_{A \in \mathcal{G}} (x-1)^{r(\mathcal{G})-r(A)} (y-1)^{n(A)} t^{f(A)}, \quad (6)$$

with  $f(A) := |\mathbf{f}^0| + |\mathbf{f}_A^{0;1}(\mathcal{E}_A)|$ .

The following proposition holds.

**PROPOSITION 3** (Operations on Tutte polynomial 2). *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two disjoint graphs with flags, then*

$$\mathcal{T}_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = \mathcal{T}_{\mathcal{G}_1} \mathcal{T}_{\mathcal{G}_2} = \mathcal{T}_{\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2}. \quad (7)$$

for any vertices  $v_{1,2}$  in  $\mathcal{G}_{1,2}$ , respectively.

**PROOF.** From Proposition 1, we must check only the behavior of the sets of flags and prove that they factorize properly. Consider  $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$  or  $\mathcal{G} = \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  their spanning c-subgraphs  $A \in \mathcal{G}$  can be simply expressed as subsets  $A = A_1 \sqcup A_2 \subset \mathcal{G}_1 \sqcup \mathcal{G}_2$  or  $A_1 \cdot_{v_1, v_2} A_2 \subset \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  where  $A_{1,2} \in \mathcal{G}_{1,2}$ . The proof is then immediate from the fact that  $\mathbf{f}_{A_1 \sqcup A_2}^0 = \mathbf{f}_{A_1 \cdot_{v_1, v_2} A_2}^0 = \mathbf{f}_{A_1}^0 \cup \mathbf{f}_{A_2}^0$  such that

$$f(A_1 \sqcup A_2) = f(A_1) + f(A_2) = f(A_1 \cdot_{v_1, v_2} A_2). \quad (8)$$

□

We are now in position to prove our first new result.

**THEOREM 1** (Contraction and cut on Tutte polynomial). *Let  $e$  be a regular edge of  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathbf{f}^0)$ . The Tutte polynomial of  $\mathcal{G}$  satisfies the relations*

$$\mathcal{T}_{\mathcal{G}} = \mathcal{T}_{\mathcal{G} \vee e} + \mathcal{T}_{\mathcal{G}/e}. \quad (9)$$

$$\mathcal{T}_{\mathcal{G} \vee e} = t^2 \mathcal{T}_{\mathcal{G}-e}. \quad (10)$$

For a bridge  $e$ , one has  $\mathcal{T}_{\mathcal{G}/e} = \mathcal{T}_{\mathcal{G}-e} = t^{-2} \mathcal{T}_{\mathcal{G} \vee e}$ , and

$$\mathcal{T}_{\mathcal{G}} = (1 + (x-1)t^2) \mathcal{T}_{\mathcal{G}/e}. \quad (11)$$

For a self-loop  $e$ , one has  $\mathcal{T}_{\mathcal{G}/e} = \mathcal{T}_{\mathcal{G}-e} = t^{-2} \mathcal{T}_{\mathcal{G} \vee e}$  and

$$\mathcal{T}_{\mathcal{G}} = (y-1 + t^2) \mathcal{T}_{\mathcal{G}-e}. \quad (12)$$

**PROOF.** The proofs of these statements are similar to the ordinary one for graph without flags. However, a special care has to be paid on the flags.

Take a regular edge  $e$ .  $\mathcal{G} \vee e$  has a set of vertex  $\tilde{\mathcal{V}} = \mathcal{V}$  and the number of connected components  $k(\mathcal{G} \vee e) = k(\mathcal{G})$ . Then, it is direct to get  $r(\mathcal{G} \vee e) = r(\mathcal{G}) = r(\mathcal{G} - e)$ . Meanwhile, contracting  $e$  in  $\mathcal{G}$  keeps the same meaning, hence  $r(\mathcal{G}/e) = (|\mathcal{V}| - 1) - k(\mathcal{G}) = r(\mathcal{G}) - 1$ . Furthermore,  $f(\mathcal{G} \vee e) = f(\mathcal{G}) + 2$  and  $f(\mathcal{G}/e) = f(\mathcal{G})$ .

We first focus on (9) and decompose the Tutte polynomial in the ordinary form as a sum on subsets containing  $e$  and a remainder that we must compare with  $\mathcal{T}_{\mathcal{G}/e}$  and  $\mathcal{T}_{\mathcal{G} \vee e}$ , respectively.

Let us choose a bijection between the two sets  $\{A \in \mathcal{G}; e \notin A\}$  and  $\{A \in \mathcal{G} \vee e\}$ . For a regular edge  $e$ ,  $A \in \mathcal{G}$ , with  $e \notin A$ , uniquely defines  $A' \in \mathcal{G} \vee e$  such that  $A' = A$  namely they both have the same vertices, edges and flags. It is also simple to check that  $\{A \in \mathcal{G} \vee e\}$  and  $\{A \in \mathcal{G} - e\}$  are in one-to-one correspondence by just removing from any  $A \in \mathcal{G} \vee e$  the two additional flags introduced by the cut of  $e$  in  $\mathcal{G} \vee e$  and get a spanning subset of  $\mathcal{G} - e$ . The mapping is clearly invertible hence bijective.

Given  $e$ , a regular edge, consider  $A \in \mathcal{G} \vee e$ , its unique corresponding element  $A'$  in  $\{B \in \mathcal{G}; e \notin B\}$  and  $A''$  its corresponding element in  $\{B \in \mathcal{G} - e\}$ . The monomial in  $\mathcal{T}_{\mathcal{G} \vee e}$  related to  $A$  can be recast in the form

$$\begin{aligned} (x-1)^{r(\mathcal{G} \vee e) - r(A)} (y-1)^{n(A)} t^{f(A)} &= (x-1)^{r(\mathcal{G}) - r(A')} (y-1)^{n(A')} t^{f(A')}, \\ &= (x-1)^{r(\mathcal{G} - e) - r(A'')} (y-1)^{n(A'')} t^{f(A'') + 2} \end{aligned} \quad (13)$$

This achieves the proof that (a)  $\mathcal{T}_{\mathcal{G} \vee e}$  and  $\sum_{A \in \mathcal{G}; e \notin A} (\cdot)$  has the same number of terms and the same monomials and (b)  $\mathcal{T}_{\mathcal{G} \vee e} = t^2 \mathcal{T}_{\mathcal{G} - e}$ .

Let us deal with the contraction now. We prove that  $\{A \in \mathcal{G}; e \in A\}$  is one-to-one with  $\{A \in \mathcal{G}/e\}$  for a regular edge  $e$ . Given a regular edge  $e$ ,  $A \in \mathcal{G}$ , with  $e \in A$ , uniquely defines  $A' \in \mathcal{G}/e$  defined by the contraction of  $e$  in  $A$  (in the sense of Definition 11).

Given  $e$  a regular edge,  $A \in \mathcal{G}/e$  and its unique corresponding subset  $A' \in \mathcal{G}$ ,  $e \in A'$ , we can write each monomial of  $\mathcal{T}_{\mathcal{G}/e}$  as

$$(x-1)^{r(\mathcal{G}/e) - r(A)} (y-1)^{n(A)} t^{f(A)} = (x-1)^{r(\mathcal{G}) - 1 - (r(A') - 1)} (y-1)^{n(A')} t^{f(A')} \quad (14)$$

which achieves the proof that  $\mathcal{T}_{\mathcal{G}/e}$  and  $\sum_{A \in \mathcal{G}; e \in A} (\cdot)$  has the same number of terms and the same terms. The proof of (9) and (10) is then complete.

We now prove  $\mathcal{T}_{\mathcal{G} - e} = \mathcal{T}_{\mathcal{G}/e}$  and (11). Given a bridge  $e$ ,  $\mathcal{G}/e = \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  and  $\mathcal{G} - e = \mathcal{G}_1 \sqcup \mathcal{G}_2$  for the maximal<sup>3</sup> subgraphs  $\mathcal{G}_i$  on each side of the bridge  $e$  with end vertices  $v_{1,2}$ . The fact that  $\mathcal{T}_{\mathcal{G} - e} = \mathcal{T}_{\mathcal{G}/e}$  follows from Proposition 3.

Second,  $\{A \in \mathcal{G}; e \notin A\}$  is one-to-one with  $\{A \in \mathcal{G}/e\}$  since  $\{A \in \mathcal{G}; e \notin A\}$  is one-to-one with  $\{A \in \mathcal{G}_1 \sqcup \mathcal{G}_2\}$  (each  $A \in \mathcal{G}$ ,  $e \notin A$  is uniquely mapped on  $A' \in \mathcal{G}_1 \sqcup \mathcal{G}_2$  by removing simply the flags issued from the cut of  $e$  in  $A$ ) and also  $\{A \in \mathcal{G}/e\} = \{A \in \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2\}$ .

Consider a bridge  $e$  in  $\mathcal{G}$  and decompose the Tutte polynomial of  $\mathcal{G}$  as

$$\begin{aligned} \mathcal{T}_{\mathcal{G}}(x, y, t) &= \sum_{A \in \mathcal{G}; e \notin A} (x-1)^{r(\mathcal{G}) - r(A)} (y-1)^{n(A)} t^{f(A)} + \mathcal{T}_{\mathcal{G}/e}(x, y, t) \\ &= \sum_{A \in \mathcal{G}/e} (x-1)^{r(\mathcal{G}/e) + 1 - r(A)} (y-1)^{n(A)} t^{f(A) + 2} + \mathcal{T}_{\mathcal{G}/e}(x, y, t) \\ &= [1 + (x-1)t^2] \mathcal{T}_{\mathcal{G}/e}(x, y, t). \end{aligned} \quad (15)$$

Furthermore, consider the bijection between  $\{A \in \mathcal{G} \vee e\}$  and  $\{A \in \mathcal{G} - e\}$  which maps any  $A \in \mathcal{G} \vee e$  uniquely on  $A' \in \mathcal{G} - e$  by removing two flags from the cut of  $e$ . Then, the monomial in  $\mathcal{T}_{\mathcal{G} \vee e}$

$$(x-1)^{r(\mathcal{G} \vee e) - r(A)} (y-1)^{n(A)} t^{f(A)} = (x-1)^{r(\mathcal{G} - e) - r(A')} (y-1)^{n(A')} t^{f(A') + 2} \quad (16)$$

yielding  $\mathcal{T}_{\mathcal{G} \vee e} = t^2 \mathcal{T}_{\mathcal{G} - e} = t^2 \mathcal{T}_{\mathcal{G}/e}$  which achieves the proof of (11).

Last, for a self-loop  $e$ , we prove (12). For a self-loop  $e$ ,  $\mathcal{T}_{\mathcal{G} - e} = \mathcal{T}_{\mathcal{G}/e}$  holds by definition. Using the bijection between  $\{A \in \mathcal{G}; e \notin A\}$  and  $\{A \in \mathcal{G} - e\}$  mapping  $A$  to  $A'$  the same subgraph without the two additional flags of  $A$  from the cut of  $e$ , and the bijection between  $\{A \in \mathcal{G}; e \in A\}$  and  $\{A \in \mathcal{G} - e\}$  given by just deleting  $e$  in  $A \in \mathcal{G}$ , we have

$$\begin{aligned} \mathcal{T}_{\mathcal{G}}(x, y, t) &= \sum_{A \in \mathcal{G} - e} (x-1)^{r(\mathcal{G} - e) - r(A')} (y-1)^{n(A')} t^{f(A') + 2} \\ &\quad + \sum_{A \in \mathcal{G} - e} (x-1)^{r(\mathcal{G} - e) - r(A')} (y-1)^{n(A') + 1} t^{f(A')} \\ &= [y - 1 + t^2] \mathcal{T}_{\mathcal{G} - e}(x, y, t). \end{aligned} \quad (17)$$

The last equality from (12) comes from the bijection between  $\{A \in \mathcal{G} \vee e\}$  and  $\{A \in \mathcal{G} - e\}$  given by removing in  $A$  the two flags coming from  $e$  and note that in  $\mathcal{T}_{\mathcal{G} \vee e}$  each monomial can be recast in the form

$$(x-1)^{r(\mathcal{G} \vee e) - r(A)} (y-1)^{n(A)} t^{f(A)} = (x-1)^{r(\mathcal{G} - e) - r(A')} (y-1)^{n(A')} t^{f(A') + 2} \quad (18)$$

<sup>3</sup>Maximal is understood in the sense that the subgraph contains all vertices, all edges and all additional flags of one bridge side.

so that, once again,  $\mathcal{T}_{\mathcal{G}_{\vee e}} = t^2 \mathcal{T}_{\mathcal{G}_{-e}}$  which achieves the proof of (12).  $\square$

Few remarks concerning this definition of Tutte polynomial for graphs with flags follow:

- For any edge  $e$ , we note that  $\mathcal{T}_{\mathcal{G}_{\vee e}} = t^2 \mathcal{T}_{\mathcal{G}_{-e}}$ .
- The contraction/cut rule seems to be the natural one in the formalism including the notion of flags.
- The polynomial  $\mathcal{T}_{\mathcal{G}}$  has always a factor of  $t^{|\mathfrak{f}^0|}$  which should be removed by a new normalization. Note that the naive prescription to introduce the variable  $t^{f(A)-f(\mathcal{G})}$  does not work. In order to remove this factor, we need another property of  $\mathcal{T}$  which follows.
- **Universality:** We know that the universal form of the Tutte polynomial (1) is given by the following statement [Theorem 2, Chap. X [5]]: *There is a unique map  $U : \mathcal{G} \rightarrow \mathbb{Z}[x, y, \alpha, \sigma, \tau]$  such that for the graph  $E_n$  made uniquely with  $n$  vertices*

$$U(E_n) = \alpha^n, \quad (19)$$

for every  $n \geq 1$  and for every  $e \in E(G)$ , we have

$$U(G) = \begin{cases} xU(G-e) & \text{if } e \text{ is a bridge,} \\ yU(G-e) & \text{if } e \text{ is a loop,} \\ \sigma U(G-e) + \tau U(G/e) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases} \quad (20)$$

Furthermore,

$$U(G) = \alpha^{k(G)} \sigma^{n(G)} \tau^{r(G)} T_G(\alpha x / \tau, y / \sigma). \quad (21)$$

The Tutte polynomial for  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  satisfies the recurrence relation given by Theorem 1 which is remarkably of the form (20). Indeed, after a change of variables as

$$\begin{cases} X = (x-1)t^2 + 1 \\ Y = y-1+t^2. \end{cases} \quad (22)$$

and given the fact that, for a given graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  and  $A \in \mathcal{G}$ ,

$$f(A) = |\mathfrak{f}^0| + 2(|\mathcal{E}| - |\mathcal{E}_A|), \quad (23)$$

we get

$$\mathcal{T}_{\mathcal{G}}(x, y, t) = t^{|\mathfrak{f}^0|} t^{2n(\mathcal{G})} T_{\mathcal{G}}(X, \frac{Y}{t^2}). \quad (24)$$

If we set  $\sigma = t^2$ ,  $\alpha = \tau = 1$  in (20),  $\mathcal{T}_{\mathcal{G}}$  is merely one solution of (20) modified by a factor  $t^{|\mathfrak{f}^0|}$  and thus defines a  $t$ -deformed version of  $T_{\mathcal{G}}$ .

- The exponent of  $t$  in  $\mathcal{T}_{\mathcal{G}}(x, y, t)/t^{|\mathfrak{f}^0|}$  is always even this is justified by the fact that each c-spanning subgraph is defined via successive cut of edges yielding each two flags. This can be easily removed by performing a change in the variable  $t \rightarrow t^{1/2}$ .
- The terminal form made with  $m$  bridges,  $n$  self-loops and  $q$  additional flags admits the Tutte polynomial

$$(1 + (x-1)t^2)^m (y-1+t^2)^n t^q = X^m Y^n t^q, \quad (25)$$

after some change of variables.

- Putting  $t = 1$  in  $\mathcal{T}_{\mathcal{G}}(x, y, t)$  yields the ordinary Tutte polynomial

$$\mathcal{T}_{\mathcal{G}}(x, y, 1) = T_{\mathcal{G}}(x, y) \quad (26)$$

obeying the ordinary contraction/deletion rule.

### 3. Ribbon graphs and Bollobas-Riordan polynomial

In this section, we first recall the generalization of the Tutte polynomial to ribbon graphs known as the BR polynomial for ribbon graphs [6, 7]. Then, we investigate ribbon with flags according to our previous developments of Subsection 2.2.



**3.1. Ribbon graphs.** We start by some basic facts on the ribbon graphs and their BR polynomial.

DEFINITION 13 (Ribbon graphs [6][15]). *A ribbon graph  $\mathcal{G}$  is a (not necessarily orientable) surface with boundary represented as the union of two sets of closed topological discs called vertices  $\mathcal{V}$  and edges  $\mathcal{E}$ . These sets satisfy the following:*

- *Vertices and edges intersect by disjoint line segment,*
- *each such line segment lies on the boundary of precisely one vertex and one edge,*
- *every edge contains exactly two such line segments.*

The previous notion of self-loops, bridges, regular edges, terminal forms (of Definition 2) can be easily reported here when the ribbon graph is seen as a simple graph namely when forgetting its ribbon character. Ribbon edges can be twisted as well. Introducing twisted edges has some consequences on the orientability of the ribbon graph. In addition, there is a new topological notion that we now describe.

DEFINITION 14 (Faces [6]). *A face is a component of a boundary of  $\mathcal{G}$  considered as a geometric ribbon graph, and hence as surface with boundary.*

If  $\mathcal{G}$  is regarded as the neighborhood of a graph embedded into a surface,  $\mathcal{F}$  is the set of faces of the embedding. A ribbon graph is denoted by  $\mathcal{G}(\mathcal{V}, \mathcal{E})$ .

DEFINITION 15 (Deletion and contraction [6]). *Let  $\mathcal{G}$  be a ribbon graph and  $e$  one of its edges.*

- *We call  $\mathcal{G} - e$  the ribbon graph obtained from  $\mathcal{G}$  by deleting  $e$  and keeping the end vertices as closed discs.*
- *If  $e$  is not a self-loop, the graph  $\mathcal{G}/e$  obtained by contracting  $e$  is defined from  $\mathcal{G}$  by deleting  $e$  and identifying its end vertices  $v_{1,2}$  into a new vertex which possesses all edges in the same cyclic order as their appeared on  $v_{1,2}$ .*
- *If  $e$  is a trivial twisted self-loop, contraction is deletion:  $\mathcal{G} - e = \mathcal{G}/e$ . The contraction of a trivial untwisted self-loop  $e$  is the deletion of the self-loop and the addition of a new connected component vertex  $v_0$  to the graph  $\mathcal{G} - e$ . We write  $\mathcal{G}/e = (\mathcal{G} - e) \sqcup \{v_0\}$ .*

The notion of contraction of a twisted or untwisted self-loop  $e$  in  $\mathcal{G}$  is subtle since it coincides with the edge deletion in the graph  $\mathcal{G}^*$  dual of  $\mathcal{G}$ , namely  $(\mathcal{G}^* - e^*)^*$  [6]. The above point on the contraction of these self-loops is compatible with this dual notion of deletion.

Spanning subgraphs in the present context keep the sense of Definition 4:  $A \in \mathcal{G}$  if  $A$  is defined by a subset of edges  $\mathcal{E}_A \subseteq \mathcal{E}$  and possesses all vertices  $\mathcal{V}$  of  $\mathcal{G}$ .

DEFINITION 16 (BR polynomial 1 [6]). *Let  $\mathcal{G}$  be a ribbon graph. We define the ribbon graph polynomial of  $\mathcal{G}$  to be*

$$R_{\mathcal{G}}(x, y, z) = \sum_{A \in \mathcal{G}} (x-1)^{r(\mathcal{G})-r(A)} (y-1)^{n(A)} z^{k(A)-F(A)+n(A)}, \quad (27)$$

where  $F(A)$  is the number of faces of  $A$ .

Notice that we use the same convention, with  $(y-1)$  parametrizing the nullity of the subgraph, as in the Tutte polynomials defined so far. This is different from the convention of BR in [6] which rather uses  $y$ . It is however simple to change variable at any moment and recover the convention used therein.

The BR polynomial obeys as well a contraction and deletion rule.

THEOREM 2 (Contraction and deletion [6]). *Let  $\mathcal{G}$  be a ribbon graph. If  $e$  is a regular edge, then*

$$R_{\mathcal{G}} = R_{\mathcal{G}/e} + R_{\mathcal{G}-e}, \quad (28)$$

for every bridge  $e$  of  $\mathcal{G}$ , one has

$$R_{\mathcal{G}} = x R_{\mathcal{G}/e}, \quad (29)$$

for a trivial untwisted self-loop

$$R_{\mathcal{G}} = y R_{\mathcal{G}-e}, \quad (30)$$

and for a trivial twisted self-loop, the following holds

$$R_{\mathcal{G}} = (1 + (y - 1)z) R_{\mathcal{G}-e}. \quad (31)$$

We emphasize that the relations (29)-(31) are useful for the determination of the BR polynomial from terminal forms which play the role of boundary conditions. The explicit formula (which has to be distinguished from the sum over spanning subgraphs) of BR polynomials of one-vertex ribbon graphs are not entirely known, to our knowledge [1]. We will only restrict to these trivial self-loops in the remaining analysis.

The following fact turns out to be crucial. During contraction and deletion moves of a regular edge, the exponent of  $z$ , namely  $k(A) - F(A) + n(A)$ , in the BR polynomial is invariant. Indeed, for a regular edge  $e$ ,  $k(A)$  does not change whether or not  $e \in A$ ;  $n(A)$ , as a difference  $|\mathcal{E}_A| - r(A)$ , is also invariant whether or not  $e \in A$ . Finally,  $F(A)$  is never affected by the contraction if  $e \in A$  whereas, if  $e \notin A$ ,  $F(A)$  remains constant after deletion of  $e$  in the graph  $\mathcal{G}$ . One could have chosen different types of exponent functions of these ingredients for defining a different polynomial invariant under contraction and deletion. However, the above choice is motivated by the fact that the combination

$$k(A) - F(A) + n(A) = 2k(A) - (|\mathcal{V}| - |\mathcal{E}_A| - F(A)) \quad (32)$$

is nothing but the genus or twice the genus (for oriented surfaces) of the subgraph  $A$ . Furthermore, writing the exponent in this form also helps for the determination of the terminal forms because  $k(A) - F(A)$  and  $n(A)$  turn out to be additive quantities with respect to the product of disjoint graphs. More motivations can be found in [6]. These remarks will be exploited in the following in order to uncover other types of BR polynomials.

**3.2. Ribbon graphs with flags.** This section starts another main result of this work. We identify an extended form of the BR polynomial for ribbon graphs with flags. If such a polynomial must be related with the polynomial by Krajewski et al. [15] introduced for the same type of graphs that mapping will be only clear after finding the multivariate form of the polynomial found in the present work and then recasting the intertwined sums of spanning subgraphs related by Chmutov duality as used in [15] in terms of a unique spanning c-subgraph sum. These subtleties will be not treated of the present work since they will not yield much insights for our final goal.

**DEFINITION 17** (Ribbon flags and external points). *A ribbon flag or half-ribbon (or half-edge or simply flag, without ambiguity) is a ribbon incident to a unique vertex by a unique segment and without forming loops. A flag has two segments one touching a vertex and another free or external segment. The end-points of any free segment are called external points of the flag.*

A ribbon flag is drawn in Figure 5.

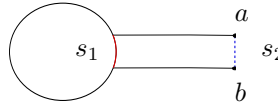


FIGURE 5. A ribbon flag  $f$  incident to one vertex disc. The two segments of  $f$ ,  $s_1$  touching the vertex and  $s_2$  a free segment with its end points  $a$  and  $b$ .

It is clear that this definition is purely combinatorial. The notion of flag can be introduced as a topological ingredient of the ribbon graph as well. Such a flag is nothing but a topological closed disc. It intersects a vertex at one segment  $s_1$  (see Figure 5). Consider on the boundary of the flag another segment  $s_2$  disjoint from  $s_1$ . The segment  $s_2$  is the external segment and it is introduced in order to distinguish the two connected segments of the boundary of the flag.

**DEFINITION 18** (Cut of a ribbon edge [15]). *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  be a ribbon graph and let  $e$  be an edge. The cut graph  $\mathcal{G} \vee e$ , is the graph obtained by removing  $e$  and let two flags attached at the end vertices of  $e$ . If  $e$  is a self-loop, the two flags are on the same vertex.*

The definitions of a ribbon graph with flags  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  and of its spanning c-subgraphs follow naturally from Definition 10 and will be refrained at this point. We denote again the spanning c-subgraph inclusion as  $A \subseteq \mathcal{G}$ .

Considered as a geometric surface, note that cutting an edge on a graph modifies the boundary faces of this graph. One can say that the new boundary faces follow the contour of the flags. But combinatorially, we introduce a discrepancy between this type of faces and the initial ones (which come from boundary of well formed edges). This will be useful in the following section dealing with the tensor situation.

**DEFINITION 19** (Closed, open faces, strands and pinching [12]). *Consider  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  a ribbon graph with flags.*

- A closed or internal face is a boundary face component of a ribbon graph (regarded as a geometric ribbon) which never passes through any free segment of additional flags. The set of closed faces is denoted  $\mathcal{F}_{\text{int}}$ . (See the closed face  $f_1$  in Figure 6)
- An open or external face is a boundary face component leaving an external point of some flag rejoining another external point. The set of open faces is denoted  $\mathcal{F}_{\text{ext}}$ . (Examples of open faces are provided in Figure 6.)
- The two boundaries lines of a ribbon edge or a flag are called strands. Each strand may belong to a closed or open face.
- The set of faces  $\mathcal{F}$  of a graph having a set of flags is defined by  $\mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$ .
- A graph is said to be open if  $\mathcal{F}_{\text{ext}} \neq \emptyset$  i.e.  $\mathfrak{f}^0 \neq \emptyset$ . It is closed otherwise.
- The pinching of a flag is a combinatorial procedure performed at one flag which identifies its external points and hence its strands.
- The pinched graph  $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{\mathfrak{f}}^0)$  of a ribbon graph  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  is a closed ribbon graph obtained after the pinching of all its flags. Thus  $\tilde{\mathcal{V}} = \mathcal{V}$ ,  $\tilde{\mathcal{E}} = \mathcal{E}$ , and  $|\tilde{\mathfrak{f}}^0| = |\mathfrak{f}^0|$  where  $\tilde{\mathfrak{f}}^0$  are now pinched flags;  $\tilde{\mathcal{F}} = \mathcal{F}_{\text{int}} \cup \mathcal{F}'$  where  $\mathcal{F}'$  are additional faces obtained from  $\mathcal{F}_{\text{ext}}$  after the pinching of  $\mathcal{G}$ .

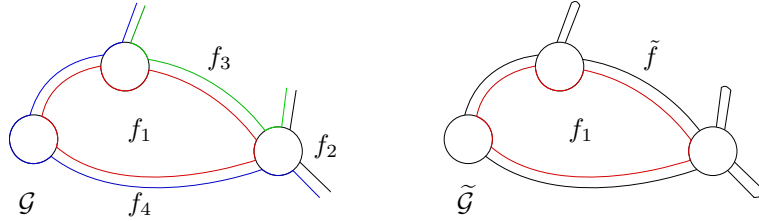


FIGURE 6. A ribbon graph  $\mathcal{G}$  with a closed face  $f_1$  (in red) and open faces  $f_{2,3,4}$  (in black, green and blue, resp.). The pinched graph  $\tilde{\mathcal{G}}$  and its two closed faces  $f_1$  and  $\tilde{f}$ .

Note that, for a graph without flags  $\mathfrak{f}^0 = \emptyset$ ,  $\mathcal{G}$  is closed and  $\tilde{\mathcal{F}} = \mathcal{F} = \mathcal{F}_{\text{int}}$  and  $\tilde{\mathcal{G}} = \mathcal{G}$ .

**DEFINITION 20** (Boundary graph [12]). • *The boundary  $\partial\mathcal{G}$  of a ribbon graph  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  is a simple graph  $\partial\mathcal{G}(\mathcal{V}_\partial, \mathcal{E}_\partial)$  in the sense of Definition 1 such that  $\mathcal{V}_\partial$  is one-to-one with  $\mathfrak{f}^0$  and  $\mathcal{E}_\partial$  is one-to-one with  $\mathcal{F}_{\text{ext}}$ . (The boundary of the graph given in Figure 6 is provided in Figure 7.)*

- The boundary of a closed graph is empty.

As a result,  $\partial\tilde{\mathcal{G}} = \emptyset$  since the pinched graph  $\tilde{\mathcal{G}}$  is closed. In practice, the boundary graph  $\partial\mathcal{G}$  can be read off the graph  $\mathcal{G}$  by inserting a vertex with degree two at each flag. External faces which are the edges of  $\partial\mathcal{G}$  are incident to these vertices. Hence, a boundary graph realizes combinatorially what is expected from the total pinching of a ribbon graph. Since  $\partial\mathcal{G}$  has only vertices with always two incident lines or one incident line if the two sides of the flag defined in fact the same external face, then it is not difficult to prove the following statement.

**PROPOSITION 4.** *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  be a ribbon graph with flags,  $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{\mathfrak{f}}^0)$  its pinching and  $\partial\mathcal{G}(\mathcal{V}_\partial, \mathcal{E}_\partial)$  its boundary. One has*

$$\mathcal{F}' = \mathcal{C}_\partial, \quad (33)$$

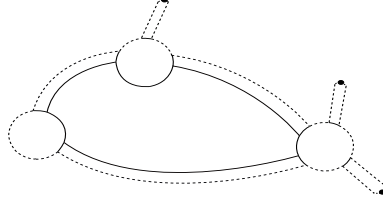


FIGURE 7. The boundary  $\partial\mathcal{G}$  of  $\mathcal{G}$  of Figure 6 is represented in dashed lines.

where  $\mathcal{F}'$  are the additional closed faces obtained by the pinching of  $\mathcal{G}$  and  $\mathcal{C}_\partial$  are the connected components of the boundary graph  $\partial\mathcal{G}$ .

The notion of edge contraction and deletion for graphs with flags can be simply understood as in Definition 15 without mentioning flags.

DEFINITION 21 (BR polynomial for ribbon graphs with flags). *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  be a ribbon graph with flags. We define the ribbon graph polynomial of  $\mathcal{G}$  to be*

$$\mathcal{R}_{\mathcal{G}}(x, y, z, s, t) = \sum_{A \in \mathcal{G}} (x-1)^{r(\mathcal{G})-r(A)} (y-1)^{n(A)} z^{k(A)-F_{\text{int}}(A)+n(A)} s^{C_\partial(A)} t^{f(A)}, \quad (34)$$

where  $C_\partial(A) = |\mathcal{C}_\partial(A)|$  is the number of connected component of the boundary of  $A$  and  $F_{\text{int}}(A) = |\mathcal{F}_{\text{int}}(A)|$ .

The polynomial  $\mathcal{R}$  (34) generalizes the BR polynomial  $R$  (27). The latter  $R$  can be only recovered from  $\mathcal{R}$  for closed ribbon graphs and at the limit  $t = 1$ . After performing the change of variable  $s \rightarrow z^{-1}$ , we are led to another extension of the BR polynomial for ribbon graphs with flags. We will refer the second polynomial to as  $\mathcal{R}'$ . In symbol, for a graph  $\mathcal{G}$ , we write

$$\mathcal{R}_{\mathcal{G}}(x, y, z, z^{-1}, t) = \mathcal{R}'_{\mathcal{G}}(x, y, z, t), \quad \mathcal{R}'_{\mathcal{G}}(x, y, z, t = 1) = R_{\widehat{\mathcal{G}}}(x, y, z), \quad (35)$$

where  $R_{(\cdot)}$  is given by Definition 16.

Graph operations  $(\mathcal{G}_1 \sqcup \mathcal{G}_2$  and  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2)$  extend to ribbon graphs [6] and ribbon graphs with flags. The product  $\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  at the vertex resulting from merging  $v_1$  and  $v_2$  (in the sense of Definition 15) respects the cyclic order of all edges and flags on the previous vertices  $v_1$  and  $v_2$ . The fact that  $R_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = R_{\mathcal{G}_1} R_{\mathcal{G}_2} = R_{\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2}$  holds for ribbon graphs without flags [6] can be extended for ribbon graphs with flags under particular conditions. The following proposition holds.

PROPOSITION 5 (Operations on BR polynomials). *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two disjoint ribbon graphs with flags, then*

$$\mathcal{R}_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = \mathcal{R}_{\mathcal{G}_1} \mathcal{R}_{\mathcal{G}_2}, \quad (36)$$

$$\mathcal{R}'_{\mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2} = \mathcal{R}'_{\mathcal{G}_1} \mathcal{R}'_{\mathcal{G}_2}, \quad (37)$$

for any disjoint vertices  $v_{1,2}$  in  $\mathcal{G}_{1,2}$ , respectively.

PROOF. Using Propositions 1 and 3, one must only check the behavior of the exponents  $k(A) - F_{\text{int}}(A) + n(A)$  and  $C_\partial(A)$ , for any  $A \in \mathcal{G}$ .

Consider first  $\mathcal{G} = \mathcal{G}_1 \sqcup \mathcal{G}_2$ . A spanning c-subgraph  $A \in \mathcal{G}$  expresses as  $A = A_1 \sqcup A_2 \in \mathcal{G}_1 \sqcup \mathcal{G}_2$  with  $A_{1,2} \in \mathcal{G}_{1,2}$ . It is straightforward to see that  $k(A)$ ,  $F_{\text{int}}(A)$ ,  $n(A)$  and  $C_\partial(A)$  are all additive quantities and therefore (36) is recovered.

Consider now  $\mathcal{G} = \mathcal{G}_1 \cdot_{v_1, v_2} \mathcal{G}_2$  and  $A = A_1 \cdot_{v_1, v_2} A_2 \in \mathcal{G}$ . We have the following relations

$$k(A) = k(A_1) + k(A_2) - 1, \quad n(A) = n(A_1) + n(A_2), \quad (38)$$

where the last equality follows from  $|\mathcal{V}(A)| = |\mathcal{V}(A_1)| + |\mathcal{V}(A_2)| - 1$ . More issues arise for the faces. Taking product at two vertices  $v_1$  of  $A_1$  and  $v_2$  of  $A_2$ , one face  $f_1$  of  $A_1$  and one face  $f_2$  of  $A_2$  enter in contact and merge. Three different situations may occur:

- Both  $f_i \in \mathcal{F}_{\text{int}}(A_i)$ ,  $i = 1, 2$ , then

$$F_{\text{int}}(A) = F_{\text{int}}(A_1) + F_{\text{int}}(A_2) - 1, \quad C_{\partial}(A) = C_{\partial}(A_1) + C_{\partial}(A_2). \quad (39)$$

-  $f_1 \in \mathcal{F}_{\text{int}}(A_1)$  and  $f_2 \in \mathcal{F}_{\text{ext}}(A_2)$  (or vice-versa), then (39) still holds because the internal face  $f_1$  is lost (becoming external by touching  $f_2$ ) whereas the connected component of the boundary graph  $\partial A_2$  which contains  $f_2$  is conserved.

- Both  $f_i \in \mathcal{F}_{\text{ext}}(A_i)$ , then

$$F_{\text{int}}(A) = F_{\text{int}}(A_1) + F_{\text{int}}(A_2), \quad C_{\partial}(A) = C_{\partial}(A_1) + C_{\partial}(A_2) - 1. \quad (40)$$

It is clear that, by changing  $s \rightarrow z^{-1}$ , the two quantities  $F_{\text{int}}(A)$  and  $C_{\partial}(A)$  add up as exponent of the  $z$  variable and always cancel the contribution from the  $k(A)$ , see (38).  $\square$

We come to the properties of  $\mathcal{R}_{\mathcal{G}}$ .

**THEOREM 3** (Contraction and cut on BR polynomial). *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  be a ribbon graph with flags. Then, for a regular edge  $e$ ,*

$$\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G} \vee e} + \mathcal{R}_{\mathcal{G}/e}, \quad (41)$$

for a bridge  $e$ , we have

$$\mathcal{R}_{\mathcal{G}} = (x - 1)\mathcal{R}_{\mathcal{G} \vee e} + \mathcal{R}_{\mathcal{G}/e}; \quad (42)$$

for a trivial twisted self-loop  $e$ , the following holds

$$\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G} \vee e} + (y - 1)z\mathcal{R}_{\mathcal{G}/e}, \quad (43)$$

whereas for a trivial untwisted self-loop  $e$ , we have

$$\mathcal{R}_{\mathcal{G}} = \mathcal{R}_{\mathcal{G} \vee e} + (y - 1)\mathcal{R}_{\mathcal{G}/e}. \quad (44)$$

**PROOF.** In the following proof, spanning c-subgraphs are simply called subgraphs. Let us make two preliminary remarks. (A) The subgraphs  $A$  of  $\mathcal{G}$  which do not contain  $e$  are precisely the subgraphs of  $\mathcal{G} \vee e$ . (B) Also, if  $e$  is not a self-loop then the map  $A \mapsto A/e$  provides a bijection from the subgraphs of  $\mathcal{G}$  which contain  $e$  to the subgraphs of  $\mathcal{G}/e$ . Note importantly that  $A$  and  $A/e$  does not have the same vertices and edges but have the same flags and so same faces.

Let us prove (41). For a regular edge  $e$ , let  $A \subseteq \mathcal{G} \vee e$ ,  $A'$  its partner in  $\mathcal{G}$  such that  $e \notin A'$  by remark (A). The fact that the monomial of  $A$  in  $\mathcal{R}_{\mathcal{G} \vee e}$  and the monomial corresponding to  $A'$  in  $\mathcal{R}_{\mathcal{G}}$  are identical is simple to check. Then  $\sum_{A \subseteq \mathcal{G}; e \notin A} (\cdot) = \mathcal{R}_{\mathcal{G} \vee e}$ .

We concentrate now on the remaining sum related to the contraction of  $e$ . In particular, we focus on sets of faces during the contraction. Choose  $A \subseteq \mathcal{G}$  with  $e \in A$  and, by remark (B), let  $A' \subseteq \mathcal{G}/e$  be its corresponding subgraph. One has

$$F_{\text{int}}(A) = F_{\text{int}}(A') \quad \text{and} \quad C_{\partial}(A) = C_{\partial}(A'). \quad (45)$$

The monomial of  $\mathcal{R}_{\mathcal{G}/e}$  related to  $A$  is of the form

$$\begin{aligned} & (x - 1)^{r(\mathcal{G}/e) - r(A)} (y - 1)^{n(A)} z^{k(A) - F_{\text{int}}(A) + n(A)} s^{C_{\partial}(A)} t^{f(A)} \\ &= (x - 1)^{r(\mathcal{G}) - 1 - (r(A') - 1)} (y - 1)^{n(A')} z^{k(A') - F_{\text{int}}(A') + n(A')} s^{C_{\partial}(A')} t^{f(A')} \end{aligned} \quad (46)$$

which achieves the proof that  $\mathcal{R}_{\mathcal{G}/e} = \sum_{A \subseteq \mathcal{G}; e \in A} (\cdot)$  and then (41) is obtained.

We prove now (42). Let  $e$  be a bridge in  $\mathcal{G}$  and decompose  $\mathcal{R}_{\mathcal{G}}$  as  $\sum_{A \subseteq \mathcal{G}; e \notin A} (\cdot) + \mathcal{R}_{\mathcal{G}/e}$ . It remains to prove that the first sum can be mapped on  $(x - 1)\mathcal{R}_{\mathcal{G} \vee e}$  but this is straightforward from  $r(\mathcal{G}) = r(\mathcal{G} \vee e) + 1$  and since all other terms remain unchanged.

The proofs of relations (43) and (44) are now provided. Consider a trivial (twisted or not) self-loop  $e$  in  $\mathcal{G}$ , then  $\sum_{A \subseteq \mathcal{G}; e \notin A} (\cdot) = \mathcal{R}_{\mathcal{G} \vee e}$  still holds in any case. Moreover, if  $e$  is a self-loop, the subgraphs of  $\mathcal{G} - e$  are one-to-one with the subgraphs of  $\mathcal{G}$  containing  $e$ . The passage from the subgraphs of  $\mathcal{G}$  containing  $e$  to those of  $\mathcal{G} - e$  or conversely is just by deleting  $e$  or gluing  $e$  to the corresponding subgraph.

Consider a trivial twisted self-loop  $e$  in  $\mathcal{G}$ . The contraction of  $e$  coincides with its deletion. Hence to each  $A \subseteq \mathcal{G}$  with  $e \in A$  and its corresponding  $A' \subseteq \mathcal{G}/e = \mathcal{G} - e$  (obtained by just

deleting  $e$  in  $A$ ), one finds that  $F_{\text{int}}(A) = F_{\text{int}}(A')$  and  $C_{\partial}(A) = C_{\partial}(A')$ . Therefore, one has the relation between the terms:

$$\begin{aligned} & (x-1)^{r(\mathcal{G})-r(A)}(y-1)^{n(A)}z^{k(A)-F_{\text{int}}(A)+n(A)}s^{C_{\partial}(A)}t^{f(A)} \\ &= (x-1)^{r(\mathcal{G}-e)-r(A')}(y-1)^{n(A')+1}z^{k(A')-F_{\text{int}}(A')+n(A')+1}s^{C_{\partial}(A')}t^{f(A')}. \end{aligned} \quad (47)$$

Thus (43) is satisfied.

Because  $e$  is a trivial untwisted self-loop, the contraction of  $e$  is its deletion supplemented by an addition of a new vertex on  $\mathcal{G} - e$  that we denote  $\mathcal{G}/e = (\mathcal{G} - e) \cup \{v_0\}$ . Given  $A \subseteq \mathcal{G}$  with  $e \in A$ , we associate a unique corresponding element  $A'$  in  $(\mathcal{G} - e) \cup \{v_0\}$  such that we delete  $e$  in  $A$  and add to it  $v_0$  as a new vertex. We can infer that  $F_{\text{int}}(A) = F_{\text{int}}(A')$ , and  $C_{\partial}(A) = C_{\partial}(A')$ . Thus, the following relation between the terms corresponding to  $A$  and  $A'$  holds:

$$\begin{aligned} & (x-1)^{r(\mathcal{G})-r(A)}(y-1)^{n(A)}z^{k(A)-F_{\text{int}}(A)+n(A)}s^{C_{\partial}(A)}t^{f(A)} \\ &= (x-1)^{r(\mathcal{G}/e)-r(A')}(y-1)^{n(A')+1}z^{k(A')-1-F_{\text{int}}(A')+n(A')+1}s^{C_{\partial}(A')}t^{f(A')}, \end{aligned} \quad (48)$$

so that (44) is obtained.  $\square$

Importantly, we cannot map  $\mathcal{R}_{\mathcal{G} \vee e}$  and  $\mathcal{R}_{\mathcal{G}/e}$  for a bridge or a trivial self-loop  $e$ . Such a mapping can be recovered only after a reduction.

**COROLLARY 1.** *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  be a ribbon graph with flags. Then, for a regular edge  $e$ ,*

$$\mathcal{R}'_{\mathcal{G}} = \mathcal{R}'_{\mathcal{G} \vee e} + \mathcal{R}'_{\mathcal{G}/e}, \quad \mathcal{R}'_{\mathcal{G} \vee e} = t^2 \mathcal{R}'_{\mathcal{G}-e}; \quad (49)$$

for a bridge  $e$ , we have  $\mathcal{R}'_{\mathcal{G}/e} = \mathcal{R}'_{\mathcal{G}-e} = t^{-2} \mathcal{R}'_{\mathcal{G} \vee e}$

$$\mathcal{R}'_{\mathcal{G}} = [(x-1)t^2 + 1] \mathcal{R}'_{\mathcal{G}/e}; \quad (50)$$

for a trivial twisted self-loop,  $\mathcal{R}'_{\mathcal{G}/e} = \mathcal{R}'_{\mathcal{G}-e} = t^{-2} \mathcal{R}'_{\mathcal{G} \vee e}$  and

$$\mathcal{R}_{\mathcal{G}} = [t^2 + (y-1)z] \mathcal{R}_{\mathcal{G}/e}, \quad (51)$$

whereas for a trivial untwisted self-loop, we have  $\mathcal{R}'_{\mathcal{G}/e} = \mathcal{R}'_{\mathcal{G}-e} = t^{-2} \mathcal{R}'_{\mathcal{G} \vee e}$  and

$$\mathcal{R}_{\mathcal{G}} = [t^2 + (y-1)] \mathcal{R}_{\mathcal{G}/e}. \quad (52)$$

**PROOF.** This is a consequence of Theorem 3, Proposition 5 and Theorem 1 and the fact that  $\mathcal{R}'_{\mathcal{G} \vee e} = t^2 \mathcal{R}'_{\mathcal{G}-e}$  that we need to detail.

The first equation of (49) needs no comment after Theorem 1. Let us prove the second equation of (49). There is a one-to-one assignment between spanning c-subgraphs  $A \subseteq \mathcal{G} \vee e$  and  $A' \subseteq \mathcal{G} - e$  given by the following:  $A'$  is obtained from  $A$  by removing the two flags issued from the cut of  $e$ . Note immediately that  $\mathcal{F}_{\text{int}}(A) \subset \mathcal{F}_{\text{int}}(A')$ . We have  $\mathcal{F}_{\text{int}}(A) \cup \mathcal{C}_{\partial}(A) = \mathcal{F}(\tilde{A})$ , where  $\tilde{A}$  is the pinched graph obtained from  $A$ . But the pinched graph  $\tilde{A}$  coincides with the pinched graph  $\tilde{A}'$  with, however, two additional flags. Indeed, they have the same vertices, edges,  $\mathfrak{f}^0(\tilde{A}) = \mathfrak{f}^0(\tilde{A}') \cup \{f_{e,1}, f_{e,2}\}$ , where  $f_{e,i}$ ,  $i = 1, 2$ , are obtained from cutting  $e$ , and for any face in  $\tilde{A}$ , one has a unique corresponding face in  $\tilde{A}'$ . This is because, given a face of  $\tilde{A}$ , either this face comes from  $\mathcal{F}_{\text{int}}(A)$  and then should be in  $\mathcal{F}_{\text{int}}(A')$ , or this face comes from  $\mathcal{C}_{\partial}(A)$  and therefore two cases may occur: (i) the removal of the flags  $f_{e,i}$  in  $A$  yields directly a closed face of  $A'$  and this before pinching  $A'$ , such that this faces belongs to  $\mathcal{F}_{\text{int}}(A')$ ; (ii) the removal of the flags  $f_{e,i}$  in  $A$  yields still an open face in  $A'$  and, so, the face belongs to  $\mathcal{C}_{\partial}(A')$ . The unicity of the face correspondence follows immediately by reversing the reasoning. Hence,  $F(\tilde{A}) = F(\tilde{A}')$  and  $f(A) = f(A') + 2$ , such that using Theorem 1 for the remaining part, one can map the terms of  $\mathcal{R}'_{\mathcal{G} \vee e}$  to those of  $t^2 \mathcal{R}'_{\mathcal{G}-e}$ .

One notices that  $\mathcal{R}'_{\mathcal{G} \vee e} = t^2 \mathcal{R}'_{\mathcal{G}-e}$  is true for any edge and not only for regular ones. For special edges,  $\mathcal{R}'_{\mathcal{G}}$  can be computed using Proposition 5 along the lines of Theorem 1 and Theorem 3. Equations in (51) and (52) simply follows from (43) and (44) of Theorem 3, respectively, and simple operations on flags as performed in Theorem 1.  $\square$

We summarize the main features of the BR polynomial for ribbon graphs with flags:

- In the ribbon formulation, once again, the cut and contraction are natural.
- $\mathcal{R}$ ,  $\mathcal{R}'$  and  $\mathcal{T}$  have similar properties concerning the number of flags  $|\mathfrak{f}^0|$  and the even exponent of the  $t$  variable.

• **Universality:** Only the polynomial  $\mathcal{R}'$  is guaranteed to have a universal property. This follows from the ordinary arguments for the universal property of the BR polynomials for closed graphs with weighted in the recurrence relation, for ordinary edges

$$\phi(\mathcal{G}) = \sigma\phi(\mathcal{G} - e) + \tau\phi(\mathcal{G}/e), \quad (53)$$

with initial conditions, for any bridge  $e$   $\phi(\mathcal{G}) = X\phi(\mathcal{G}/e)$  and the data of all BR polynomials for one-vertex ribbon graphs [6]. Given (49), once again by performing some change of variables, we can recast the relations satisfied by  $\mathcal{R}'$  in this form.

- The polynomial  $\mathcal{R}'$  evaluated on terminal forms made with  $p$  bridges,  $m$  trivial untwisted self-loops,  $n$  twisted self-loops and  $q$  flags is easily computable as

$$[(x-1)t^2 + 1]^p [t^2 + y - 1]^m [t^2 + (y-1)z]^n t^q = X^p Y^m Z^n t^q. \quad (54)$$

- Note that, although there is a one-to-one map between  $\{A \subseteq \mathcal{G} \vee e\}$  and  $\{A' \subseteq \mathcal{G} - e\}$  (see in the proof above),  $A'$  has the same vertices and edges but different flags and so different faces. There is *a priori* no relation between  $\mathcal{R}_{\mathcal{G} \vee e}$  and  $\mathcal{R}_{\mathcal{G} - e}$  due to the discrepancy between their number of internal faces and boundary components. A relation between the polynomial of  $\mathcal{G} \vee e$  and  $\mathcal{G} - e$  can be only recovered for the  $\mathcal{R}'$  polynomial. For any edge  $e$ ,  $\mathcal{R}'_{\mathcal{G} \vee e} = t^2 \mathcal{R}'_{\mathcal{G} - e}$ .

- We have

$$\mathcal{R}_{\mathcal{G}}(x, y, z, z^{-1}, t) = \mathcal{R}'_{\mathcal{G}}(x, y, z, t), \quad \mathcal{R}'_{\mathcal{G}}(x, y, z, t = 1) = \mathcal{R}_{\mathcal{G}}(x, y, z), \quad (55)$$

hence  $\mathcal{R}_{\mathcal{G}}$  and  $\mathcal{R}'_{\mathcal{G}}$  are generalized BR polynomials satisfying contraction and deletion operations. Finally, to recover the Tutte polynomial we set

$$\mathcal{R}_{\mathcal{G}}(x, y, z = 1, s = 1, t) = \mathcal{T}_{\mathcal{G}}(x, y, t). \quad (56)$$

from which the ordinary Tutte polynomial  $T_{\mathcal{G}}$  is directly inferred.

#### 4. Rank $D$ stranded graphs with flags and a generalized polynomial invariant

This section undertakes the definition of a new polynomial for particular graphs which aims at generalizing the BR polynomial  $\mathcal{R}$  for graphs on surfaces with boundaries in its most expanded form as treated in the Section 3.2. The central notion of graphs discussed below is combinatorial and can be always pictured in a 3D space. Our main result appears in Theorem 4 after a thorough definition of the type of graphs extending ribbon graphs with flags for which a generalized polynomial invariant turns out to exist.

The primary notion of rank  $D$  colored tensor graphs considered here has been introduced by Gurau in [10]. As ribbon graphs can be dually mapped onto triangulations of surfaces in 2D, colored tensor graphs can be interpreted as simplicial complexes or dual of triangulations of topological spaces in any dimension. They are of particular interest in certain quantum field theoretical frameworks defined with tensor fields hence the name tensor graphs<sup>4</sup>. The importance of these graphs has been highlighted by Gurau which proved that the cellular structure associated with these colored graphs generates dually only simplicial pseudo-manifolds of  $D$  dimensions [11]. It has been also proved that a colored tensor graph which is bipartite induces naturally an orientation of the dual complex [8].

Some anterior studies have addressed the extension of the BR polynomial for higher dimensional object within the framework of such graphs. Mainly, two authors Gurau [12] and Tanasa [18] have defined two separate notions of extended BR polynomial. Let us do a blitz review of their results and compare these to the one obtained in the present work.

Gurau defined a multivariate polynomial invariant for colored tensor graphs dually associated with simplicial complexes with boundaries for any dimension  $D$ . The polynomial that we obtain in the present work admits a multivariate form which is related to the polynomial obtained by

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<sup>4</sup>Ribbon graphs are, in this sense, rank 2 or matrix graphs.

this author restricted to 3D but extends it to a larger type of graph than only the colored tensor graphs. We emphasize that the difficulties encountered by Gurau (as well as Tanasa, see below) for defining a contraction procedure for such type of graphs without destroying the entire graph structure will be improved our scheme.

In a different perspective, the work by Tanasa in [18] deals with tensor graphs without colors which are equipped with another vertex. The polynomial as worked out by this author is only valid for graphs triangulating topological objects without boundary. The polynomial that we define is radically different from that one in several features, since mainly, it relies on the presence of colors in the graph.

**4.1. Stranded, tensor, colored graphs with flags.** After some operations, the initial type of rank  $D$  colored tensor graphs generates a final class of graphs which will be the one of interest in our analysis. Like ribbon graphs, colored tensor graphs have both a topological meaning (realized in the dual triangulation) and a combinatorial formulation that will be our main concern here. Before defining colored tensor graphs, let us introduce the combinatorial concept of stranded graph structure, the true backbone of this theory.

**Stranded and tensor graphs.** We start by primary notions of stranded structures and graphs.

**DEFINITION 22** (Stranded vertex and edge). • *A rank  $D$  stranded vertex is a collection of segments called bows or strands which can be drawn in a 3-ball hemisphere (or a spherical cup in a three dimensional space) such that:*

- (a) *the bows are not intersecting;*
- (b) *all end points of the bows are drawn on the boundary (circle) of the equatorial disc (or section) called the vertex frontier;*
- (c) *these end points can be partitioned in sets called pre-flags with  $0, 1, 2, \dots, D$  elements;*
- (d) *the pre-flags should form a connected collection that is each pre-flag should be connected to any other pre-flag by a tree of bows.*

*The coordination (or valence or degree) of a rank  $D$  stranded vertex is the number of its pre-flags. By convention, (C1) we include a particular vertex made with one disc and assume that it is a stranded vertex of any rank made with a unique closed strand and (C2) a point is a rank 0 stranded vertex.*

• *A rank  $D$  stranded edge is a collection of segments called again strands such that:*

- (a') *the strands are not intersecting (but can cross, i.e. can be non parallel);*
- (b') *the end points of the strands can be partitioned in two disjoint parts called sets of end segments of the edge such that a strand cannot have its end points in the same set of end segments;*
- (c') *the number of strands is  $D$ .*

Rank  $D > 0$  stranded vertices just enforce that any entering strand in the vertex should be exiting by another point at the frontier of the vertex. According to our convention, the ordinary disc and ribbon edges of ribbon graphs are valid stranded vertex and stranded edges.

The notion of connectivity should be clarified. Vertices (with connected pre-flags) and edges are by convention connected objects even though graphically they sometimes appear disconnected in the ordinary topological sense. This is will be justified later on based on the dual that can represent such objects after few more constraints. Examples of stranded vertices and edges with rank  $D = 4$  and  $D = 5$ , respectively, have been provided in Figure 8.

**DEFINITION 23** (Stranded and tensor graphs). • *A rank  $D$  stranded graph  $\mathcal{G}$  is a graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  which admits:*

- (i) *rank  $D$  stranded vertices;*
- (ii) *rank at most  $D$  stranded edges;*
- (iii) *One vertex and one edge intersect by one set of end segments of the edge which should coincide with a pre-flag at the frontier vertex. All intersections of vertices and edges are pairwise distinct.*



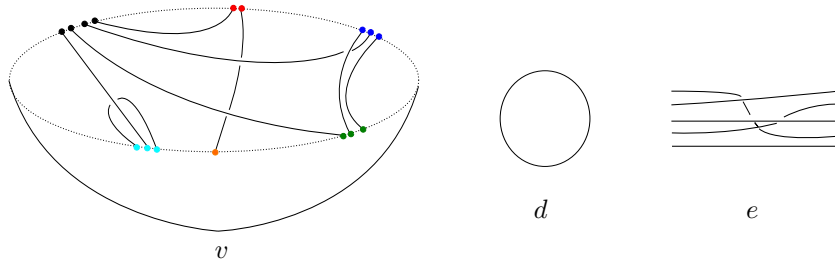


FIGURE 8. A rank 4 stranded vertex  $v$  of coordination 6, with connected pre-flags (highlighted with different colors) with crossing bows; a trivial disc vertex  $d$ ; a rank 5 edge  $e$  with non parallel strands.

- A rank  $D$  tensor graph  $\mathcal{G}$  is a rank  $D$  stranded graph such that:
  - (i') the vertices of  $\mathcal{G}$  have a fixed coordination  $D + 1$  and their pre-flags have a fixed cardinal  $D$ . From the point of view of the pre-flags, the pattern followed by each stranded vertex is that of the complete graph  $K_{D+1}$ ;
  - (ii') the edges of  $\mathcal{G}$  are of rank  $D$  and their strands are parallel.

Some illustrations of a rank 3 stranded and tensor graphs are given in Figure 9 and 10, respectively.

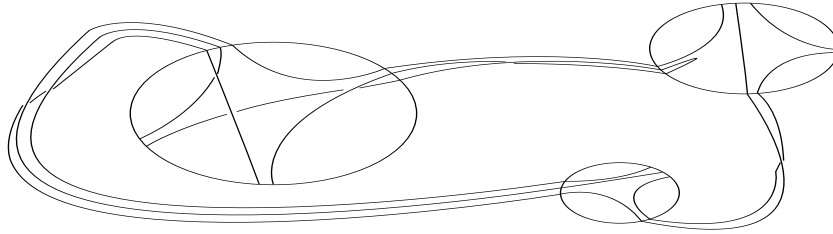


FIGURE 9. A rank 3 stranded graph.

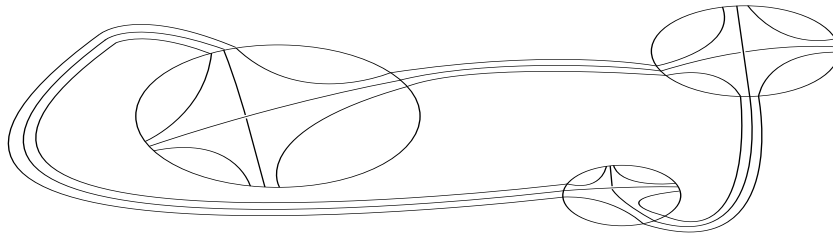


FIGURE 10. A rank 3 tensor graph with rank 3 vertices as with fixed coordination 4, pre-flags with 3 points linked by bows according to the pattern of  $K_4$ ; edges are rank 3 with parallel strands.

It should be pointed out that, although the vertices and edges of stranded and tensor graphs are drawn in a three dimensional space, we do not treat these as embedded graphs. The reason for this follows. There is an equivalent way to construct rank  $D$  stranded vertices, edges and graphs using uniquely combinatorial (permutation) maps [19] (see Chap. X). For this we need to label each pre-flag point (or cross in the jargon of [15]) and define to which it is associated. There should be  $\sigma_0$  a vertex map,  $\sigma_1$  an edge map and  $\theta$  a pre-flag map partitioning the set of crosses. For simple and ribbon graphs such permutation triple exists [19]. For instance, they have

been extensively used in [15] in order to extend Chmutov duality to ribbon graphs with flags. At this moment, there is no equivalent formulation for higher rank stranded graphs in the way we introduce it here. However, one realizes that these maps can be defined point by point for each graph constituent. In other words,  $\sigma_0$  should be defined for each vertex,  $\theta$  for each pre-flag,  $\sigma_1$  for each edge. Remark that to capture the general features that these maps should satisfy for stranded or even tensor graphs can be a non trivial task.

At the end, we will always consider as identical two rank  $D$  stranded graphs which are defined by the same  $(\sigma_0, \theta, \sigma_1)$ . Examples of identical vertices and edges have been given in Figures 11 and 12. Thus we can modify a stranded graph and still obtain the same graph after moving the pre-flags at the frontier vertex (the order of their points can be also changed), deform and cross arbitrarily bows and strands. The sole point to be required is to keep the incidence relations of strand segments between the labeled points. For these reasons, henceforth, we will always use a “minimal” graphical representation for stranded vertices and edges which is the one defined by strands using the “shortest” path between the pre-flags points.

A side remark is that, in Definition 23, the precision in (ii') that the strands of edges in tensor graphs are parallel seems now superfluous since, using the above combinatorial maps, we can always make parallel non parallel strands. The price to pay for satisfying that is to get a more peculiar vertex with more bow crossings following the fact that we should preserve the same incidence of bows. Hence, the word parallel in (ii') is seen as the simplest choice possible for edges making as simple as possible the vertex in the case of tensor graphs.

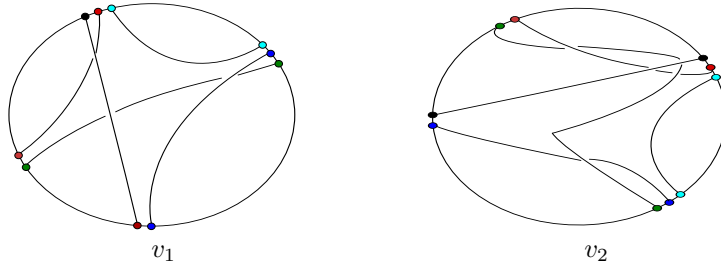


FIGURE 11. Two identical vertices,  $v_1$  and  $v_2$ , defined up to a rotation of their frontier; note that the pairing of pre-flag points (labeled by colors here) is the same (deformation and crossing of their bows are meaningless).  $v_2$  will be not preferred though because one of its bows, the one between the two green pre-flag points, do not use the shortest path between pre-flags points.

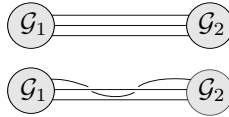


FIGURE 12. Two identical rank 3 edges between two graphs (shaded).

Given a stranded graph and collapsing its stranded vertices to points and its edges to simple lines, one obtains a simple graph in the sense of Definition 1. This justifies the fact that stranded graphs are graphs. Moreover, it is with respect to this simple graph point of view that we will say that a stranded graph is connected or not. In other words, a stranded graph is connected if its corresponding collapsed graph is connected. For instance, both graphs of Figure 9 and 10 are connected.

We denote, when no confusion is possible,  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  a rank  $D$  stranded graph with vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$ . It can be seen that any rank  $D$  stranded graph is a rank  $D'$  stranded graph, with  $D' \geq D$ . This is however not true for tensor graphs. Calling a rank  $D$  stranded graph in the remaining will often refer to the minimal rank for which this graph is well defined.

**Lower rank stranded graphs.** Let us give low rank examples of the above graphs which turns out to be either trivial or known.

- (0) Rank 0 stranded graphs are simple graphs made with only points as vertices and no edges.
- (1) Rank 1 stranded graphs as for vertices segments (note that any pre-flag reduces to a point and, in order to form a connected collection, one has to consider a unique segment between two pre-flags) and for edges segments as well. Such rank 1 stranded graphs cannot be directly identified with simple graphs in the sense of Definition 1. They can be viewed as simple graphs if one collapses all vertices to points. Hence, neither rank 0 nor rank 1 stranded graphs allows to define simple graphs in general. In fact, for any rank  $D$ , the simple graph structure cannot be directly achieved without using the collapsing procedure described above.
- (2) General ribbon graphs, in the sense of Definition 13, are one-to-one with particular rank 2 stranded graphs. Given a ribbon graph  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  (Definition 13) and an edge  $e$  of  $\mathcal{G}$ . The ribbon edge  $e$  touches its end vertex or vertices in two segments  $s$  and  $s'$ . Consider the two distinct segments on the boundary of  $e$  which are not  $s$  and  $s'$  (denoted by  $s_1$  and  $s_2$  in Figure 13A). These two segments define the two (not intersecting and potentially twisted) strands of a rank 2 stranded edge with  $s$  and  $s'$  as end segments. By convention, a vertex with no incident line is a stranded vertex. So it is a stranded graph of any rank in particular  $D = 2$ . Consider now a vertex  $v$  of  $\mathcal{G}$  with incident edges  $e_1, e_2, \dots, e_p$ ,  $p \in \mathbb{N}$ . We can construct a stranded vertex  $v'$  of rank 2 in the following manner. The frontier vertex of  $v'$  is simply the boundary of  $v$ . The end points of end segments of  $e_k$ ,  $1 \leq k \leq p$ , define the pre-flags (with exactly 2 points) of  $v'$ . From these pre-flags, one defines the bows of  $v'$  as the segments between the end segments of  $e_k$  which lies in  $v$ . See Figure 13B.

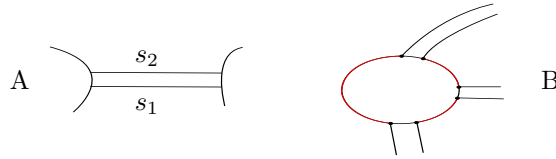


FIGURE 13. Ribbon edge and vertex of a ribbon graph as a rank 2 stranded edge and vertex of a stranded graph.

Ribbon graphs equipped with vertices with fixed coordination equals to 3 are rank 2 tensor graphs. The reason why the converse is not true is that, in any rank 2 tensor graph, the bows might be not cyclically disposed on the vertex frontier. Thus the definition of any ribbon graph vertex as a disc may be not always satisfied.

While a stranded graph can be regarded as an abstract object, tensor graphs actually possess a topological content.

**DEFINITION 24** (Dual of a tensor graph). *Let  $\mathcal{G}$  be a rank  $D$  tensor graph. A vertex of  $\mathcal{G}$  represents a  $D$  simplex and an edge of  $\mathcal{G}$  represents a  $(D - 1)$  simplex. The graph  $\mathcal{G}$  is dual to a simplicial complex obtained from the gluing of  $D$  simplexes along  $(D - 1)$  simplexes lying at their boundary.*

As an illustration of the above definition, a rank 3 tensor graph represents a simplicial complex in 3D composed by tetrahedra (3-simplexes) which are glued along their boundary triangles (2-simplexes). For rank  $D$  tensor graph, a pre-flag is of cardinal  $D$ . Each couple of points induces a segment in the pre-flag, we obtain  $D(D - 1)/2$  disjoint segments. These pre-flag segments are graphically pictured by  $D$  points or  $D - 1$  joined (adjacent) segments between these  $D$  points at the frontier vertex. One may interpret that these  $D(D - 1)/2$  disjoint segments as the number of  $(D - 2)$ -simplexes of the boundary of a  $D - 1$  simplex. For  $D = 3$ , there are  $D(D - 1)/2 = 3$

edges in a triangle. Combinatorially, we draw these edges as  $D = 3$  points or  $D - 1 = 2$  joined segments.

We now come back to our previous convention that stranded vertices and edges are connected. Because the dual of these two entities are connected topological simplexes only under tensor graph axioms, we simply assume that, in tensor graphs and by extension in stranded graphs, these objects are also connected.

**DEFINITION 25 (Rank  $D$  flag).** *A rank  $D$  stranded flag (or half-edge or simply flags, when no confusion may occur) is a collection of  $D$  parallel segments called strands satisfying the same properties of strands of rank  $D$  edges but the flag is incident to a unique rank  $D'$  stranded vertex, with  $D' \geq D$ , by one of its set of end segments without forming a loop.*

*A rank  $D$  flag has two sets of end segments: one touching a vertex and another called free or external set of end segments, the elements of which called themselves free or external segments. The  $D$  end-points of all free segments are called external points of the rank  $D$  flag. (See Figure 14.)*

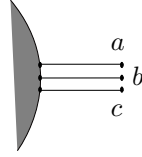


FIGURE 14. A rank 3 stranded flag with its external points  $a$ ,  $b$  and  $c$ .

**DEFINITION 26 (Cut of an edge).** *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  be a rank  $D$  stranded graph and  $e$  a rank  $d$  edge of  $\mathcal{G}$ ,  $1 \leq d \leq D$ . The cut graph  $\mathcal{G} \vee e$  or the graph obtained from  $\mathcal{G}$  by cutting  $e$  is obtained by replacing the edge  $e$  by two rank  $d$  flags at the end vertices of  $e$  and respecting the strand structure of  $e$ . (See Figure 15.) If  $e$  is a self-loop, the two flags are on the same vertex.*

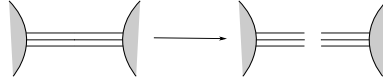


FIGURE 15. Cutting a rank 3 stranded edge.

The cut graph  $\mathcal{G} \vee e$  is called a rank  $D$  stranded graph with flags. The presence of flags is compatible with rank  $D$  stranded vertices and edges. Furthermore, in a stranded graph, we will always consider that all pre-flags are used. In other words, all pre-flags are necessarily either in contact with edges or in contact with flags. An example of such a graph is provided in Figure 16.

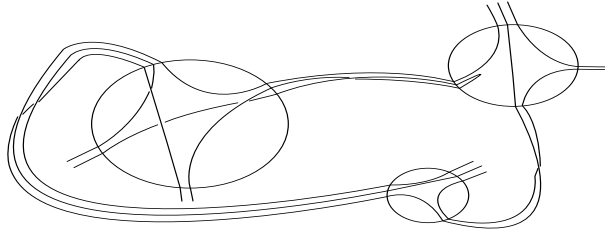


FIGURE 16. A rank 3 stranded graph with all pre-flags used.

Hence, given a set of additional flags  $\mathfrak{f}^0$  in some stranded graph, assuming that  $\mathfrak{f}^0 = \emptyset$  simply means that all pre-flags are in contact with edges. Any rank  $D$  stranded graph with additional set of flag  $\mathfrak{f}^0$  is denoted  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$ .

The notion of c-subgraph and spanning c-subgraph follows naturally from Definition 10 with however the new meaning of the cut operation. The spanning c-subgraph inclusion will be again denoted by  $A \subseteq \mathcal{G}$ . To keep track of the internal structure of the vertices, c-subgraphs will be preferred to subgraphs which adds to the fact that we will only use stranded graphs with all pre-flags fully connected to flags. This also justifies why in the formulation of [12] the deletion an edge is in fact the cut of the edge in the sense precisely defined above. Thus, the cut operation should be the natural operation within this framework.

Particular edges as self-loops, bridges (defined through a cut of an edge which brings an additional connected component in the graph), regular edges and terminal forms are extended in the present context because the stranded graph admits a graph structure.

Most of the concepts of Definition 19 are valid for stranded graphs. Faces of a stranded graph  $\mathcal{G}$  are the maximal connected components made with strands of  $\mathcal{G}$ . Closed faces never pass through any free segment of additional flags.  $\mathcal{F}_{\text{int}}$  denotes the set of these closed faces. Open or external faces are maximally connected component leaving an external point of some flag rejoining another external point. The set of open faces is denoted  $\mathcal{F}_{\text{ext}}$ . As usual, the set of faces  $\mathcal{F}$  of a graph is defined by  $\mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$ . Open and closed stranded graphs follows the same ideas as in Definition 19 as well.

Several notions (such as open and closed face) which are totally combinatorial in the stranded situation bear a true topological content in the tensor graph case. For instance, a closed face in a rank 3 tensor graph is a 2D surface in the bulk (interior) of the dual triangulation. An open face is a surface intersecting the boundary of the simplicial complex dual to the graph.

Let us discuss in more details the structure of an edge  $e$ . Assuming that  $e$  is of rank  $d$ , consider the two pre-flags  $f_1$  and  $f_2$  where  $e$  is branched to its end vertex  $v$  (a self-loop situation) or vertices  $v_{1,2}$  (a non self-loop case). It may happen that, after branching  $e$ ,  $p$  closed faces are formed<sup>5</sup> such that these closed faces are completely contained in  $e$  and  $v$  or  $v_{1,2}$ . These closed faces will be called **inner faces** of the edge. An edge with  $p$  inner faces is called  **$p$ -inner edge**. See Figure 17 A,B and C for an illustration in the rank 3 situation. The remaining strands passing through  $f_1$  and  $f_2$  which are not used in the inner faces are called **outer strands**. They connect  $f_1$  and  $f_2$  to other pre-flag families  $\{f_{1,i}\}$  and  $\{f_{2,j}\}$ , called **neighbors** of  $f_1$  and  $f_2$  respectively (this has been also illustrated in Figure 17). Dealing with a self-loop, we add the condition that neighbor families should satisfy  $f_1 \notin \{f_{2,j}\}$  and  $f_2 \notin \{f_{1,i}\}$ . Note that these two families may have pre-flags in common for a self-loop situation. If one of the neighbor families is empty, it means that for a non self-loop case,  $f_1$  or  $f_2$  define a vertex. For the self-loop case, the families can be empty but then there should exist some bows between  $f_1$  and  $f_2$  and, if both families are empty,  $f_1$  and  $f_2$  define together a vertex.

For any type of edge,

- Assuming that  $\{f_{1,i}\}$  and  $\{f_{2,j}\}$  are not empty, this implies that the outer strands of  $f_1$  and  $f_2$  are related by strands via  $e$ . Consequently,  $\{f_{1,i}\}$  and  $\{f_{2,j}\}$  are connected via strands in  $e$ .
- Assuming that  $\{f_{1,i}\}$  is empty but  $\{f_{2,j}\} \neq \emptyset$  (or vice-versa without loss of generality) the outer strands of  $f_2$  are all related via  $e$  (and in the case of a self-loop, these outer strands may be related with bows in common between  $f_1$  and  $f_2$ ). Therefore  $\{f_{2,j}\}$  is connected via strands in  $e$ .
- Assuming that both  $\{f_{1,i}\}$  and  $\{f_{2,j}\}$  are empty, then  $e$  generates only inner faces.

The notion of edge contraction can be defined at this point.

**DEFINITION 27** (Stranded edge contractions). *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  a rank  $D$  stranded graph with flags. Let  $e$  be a rank  $d$  edge with pre-flags  $f_1$  and  $f_2$  and consider their neighbor families  $\{f_{1,i}\}$  and  $\{f_{2,j}\}$ .*

- *If  $e$  is a rank  $d$   $p$ -inner edge but not a self-loop, the graph  $\mathcal{G}/e$  obtained by contracting  $e$  ‘softly’ is defined from  $\mathcal{G}$  by removing the  $p$  inner faces generated by  $e$ , replacing  $e$  and its end vertices  $v_{1,2}$  by  $p$  disjoint disc vertices and a new vertex  $v'$  which possesses all edges but  $e$  and all flags in the same cyclic order as their appear on  $v_1$  and  $v_2$ , and strands obtained by connecting*

---

<sup>5</sup>If  $e$  is a self-loop,  $p \leq d$ ; if  $e$  is not a self-loop,  $p \leq d/2$  if  $d$  is even or  $p \leq (d-1)/2$ , if  $d$  is odd.

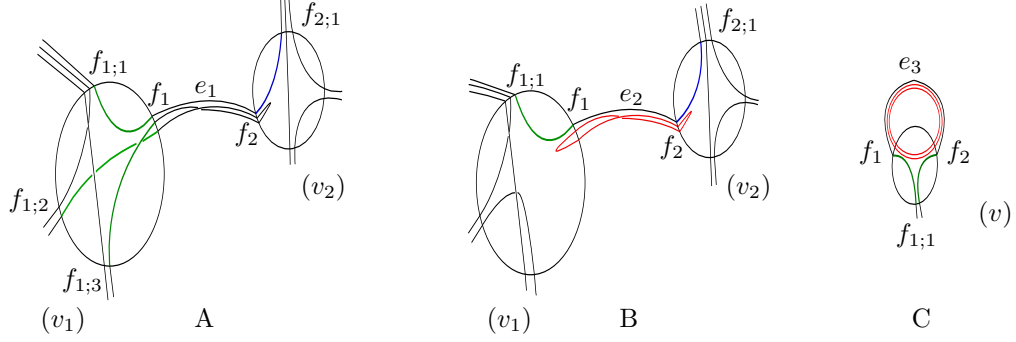


FIGURE 17. Some rank 3  $p$ -inner edges: 0-inner  $e_1$  (A), 1-inner  $e_2$  (B) and 2-inner  $e_3$  (C) edges (with inner faces highlighted in red). Outer strands highlighted (in green and blue) for each  $p$ -inner edge case of A, B, C and their attached neighbor pre-flags.

directly  $\{f_{1,i}\}$  and/or  $\{f_{2,j}\}$  via the outer strands of  $f_1$  and  $f_2$ . (See examples of rank 3 edge contraction in Figure 18 A' and B'.)

- If  $e$  is a rank  $d$   $p$ -inner self-loop, the graph  $\mathcal{G}/e$  obtained by contracting  $e$  ‘softly’ is defined from  $\mathcal{G}$  by removing the  $p$  inner faces of  $e$ , by replacing  $e$  and its end vertex  $v$  by  $p$  disjoint disc vertices and one vertex  $v'$  having all edges as  $v$  but  $e$  and all flags and strands built in the similar way as previously done by connecting the neighbor families of  $f_1$  and/or  $f_2$ . (See an example of rank 3 self-loop contraction in Figure 18C'.)

If there is no outer strand left after removing the  $p$  inner faces of  $e$  then the vertex  $v'$  is empty.

- For any type of  $p$ -inner edges, the graph  $(\mathcal{G}/e)^{\sharp}$  obtained by contracting  $e$  ‘hardly’ is defined in the same way as above for each type of edge but without adding the extra  $p$ -discs.

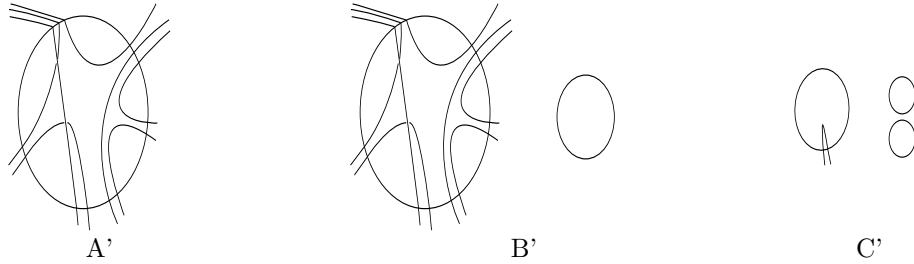


FIGURE 18. Graphs A', B' and C' obtained after edge contraction of A, B and C of Figure 17, respectively.

Soft and hard contractions agree obviously on 0-inner edges. The result of a hard contraction can be always inferred from that one of a soft contraction performed on the same graph by simply removing the extra discs introduced at the end of the soft procedure. Introducing this discrepancy in the edge contraction will be justified in the following.

The terms ‘cyclic order’ can be removed from the definition provided we keep track of the pairing of the pre-flags points. The cyclic order preservation is a simple and convenient choice for recovering the final vertex after the contraction.

One may directly check that contracting a non self-loop in a ribbon graph can be immediately seen as a rank 2 edge (soft or hard) contraction. For self-loop in ribbon graphs, we conjecture that the soft contraction procedure agrees with ordinary contraction of ribbon self-loop once again. In fact, one may check that a trivial self-loop contraction for a ribbon graph coincides with the notion contraction of a  $p$ -inner self-loop contraction in the soft sense of Definition 27, with  $p = 0, 1, 2$ ,

when the ribbon graph is viewed as a rank 2 stranded graph<sup>6</sup>. For a general self-loop  $e$ , we have strong hints that the soft contraction still coincides with the simple contraction of  $e$  in a ribbon graph  $\mathcal{G}$  defined through the dual graph  $(\mathcal{G}^* - e^*)^* = \mathcal{G}/e$  [6] hence, our above conjecture.

**PROPOSITION 6.** *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  a rank  $D$  stranded graph with flags. Let  $e$  be a rank  $d$  edge. The graphs  $\mathcal{G}/e$  and  $(\mathcal{G}/e)^\natural$  obtained by soft or hard contraction of  $e$ , respectively, admit a rank  $D$  stranded structure.*

**PROOF.** The following does not depend on the type of contraction. After contraction of an edge  $e$ , remark that the remaining edges in the graph  $\mathcal{G}$  are untouched. Hence they are at most rank  $D$  stranded edges. The gluing of these edges and vertices of the  $\mathcal{G}/e$  is unchanged as well. Ignoring the potentially present  $p$  discs which are rank  $D$  stranded graphs for any  $D$ , the unique issue that we need to check is whether or not the vertex  $v'$  obtained after edge contraction admits a rank  $D$  stranded structure.

To proceed with this, observe first that bows in  $v'$  are defined to be previous non intersecting bows or merged bows from neighbors pre-flags. Obviously all these bows are not intersecting before and after contraction. The frontier vertex can be easily extended (by deformation) to contain all pre-flags of the vertex  $v_1$  and  $v_2$  except  $f_1$  and  $f_2$  the pre-flags of  $e$ . The key point is that the vertex  $v'$  may be disconnected with respect to its pre-flags. One must check that every connected component in  $v'$  are rank  $D$  stranded vertices.

The remaining point is then quickly solved from the fact that before and after contraction all points are connected via bows in the graph. This means that, before contraction, all pre-flags have all of their points connected by bows to other pre-flag points. After contraction, some of the bows are removed. But whenever a bow is removed its end points are removed as well. Thus all remaining bows (if exist) linking all pre-flags are either untouched or carefully reconnected between themselves in the neighbor families of  $f_1$  and/or  $f_2$  (if exist). Hence there is no point in any remaining pre-flags without bow attached. Now if the vertex  $v'$  gets disconnected and some of the pre-flags form a separate set, all points of these pre-flags must be fully connected between themselves by bows. This ensures that any connected set of pre-flags forms a new stranded vertex with rank at most  $D$ . □

After the above discussion, the following proposition is straightforward.

**PROPOSITION 7 (Rank evolution under contraction).** *Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  a rank  $D$  stranded graph with flags. Let  $e$  be a rank  $d$  edge. The ranks of the graphs  $\mathcal{G}/e$  and  $(\mathcal{G}/e)^\natural$  are less than or equal to  $D$ .*

At this point, one may wonder if the additional discs obtained after soft edge contraction are not inessential features of the graphs for the remaining analysis. In fact, they will be very useful for the preservation of number of faces during the edge contraction of the graph as it is the case for ribbon graphs. Furthermore as discussed above, if one wants to match with the definition of a deletion in the dual, they could become totally relevant. Hard contractions are also useful but in another context, for instance, contracting particular tensor subgraphs called dipoles (see for instance [4]). We will however focus only soft contractions. What we shall need is just to partially remove the disc which could be generated and work “up to trivial discs” hence in the framework of equivalence class of graphs.

**DEFINITION 28 (Graph equivalence class).** *Let  $D_{\mathcal{G}}$  be the subgraph in a rank  $D$  stranded graph  $\mathcal{G}$  defined by all of its trivial disc vertices and  $\mathcal{G} \setminus D_{\mathcal{G}}$  the rank  $D$  stranded graph obtained after removing  $D_{\mathcal{G}}$  from  $\mathcal{G}$ .*

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<sup>6</sup> Indeed, for a ribbon graph, a trivial untwisted self loop is equivalent to a 1- or 2-inner self-loop. It is a 1-inner self-loop if one of the face of the self-loop does not immediately close in the vertex and pass through another edge. A trivial untwisted self-loop with one vertex and one edge is a 2-inner self-loop. A trivial twisted self-loop can be 1- or 0-inner self-loop, depending on the number of edges of the vertex having the self-loop. If there is more than one edge in the vertex, it is a 0-inner self-loop, otherwise it is a 1-inner self-loop. Treating systematically these configurations of trivial self-loops in a ribbon graph, one can prove the above claim for these cases.

Two rank  $D$  stranded graphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are “equivalent up to trivial discs” if and only if  $\mathcal{G}_1 \setminus D_{\mathcal{G}_1} = \mathcal{G}_2 \setminus D_{\mathcal{G}_2}$ . We note  $\mathcal{G}_1 \sim \mathcal{G}_2$ .

One can check that  $\sim$  is indeed an equivalence relation. The next important notion to identify is that of (spanning) c-subgraphs of an equivalence class  $\mathfrak{G}$  of a graph  $\mathcal{G}$ . This can be done as follows. A (spanning) c-subgraph  $\mathfrak{g}$  of  $\mathfrak{G}$  is the equivalence class of a (spanning) c-subgraph of any of its representative. To see why this concept is independent of the representative holds by definition: any (spanning) c-subgraphs of some representative coincides with some (spanning) c-subgraphs of any another representative up to discs. The fact that one c-subgraph is spanning in some  $\mathcal{G}$  means that it includes all vertices of  $\mathcal{G}$ . Removing all discs, it becomes spanning in  $\mathcal{G} \setminus D_{\mathcal{G}}$ , for any  $\mathcal{G}$ .

**Colored tensor graphs.** Apart from the stranded structure, the second important feature of the type of graphs which we will study is the coloring.

**DEFINITION 29** (Colored and bipartite graphs). • A  $(D+1)$  colored graph is a graph together with an assignment of a color belonging to the set  $\{0, \dots, D\}$  to each of its edges such that no two adjacent edges share same color.

• A bipartite graph is a graph whose set  $\mathcal{V}$  of vertices is split into two disjoint sets, i.e.  $\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-$  with  $\mathcal{V}^+ \cap \mathcal{V}^- = \emptyset$ , such that each edge connects a vertex  $v^+ \in \mathcal{V}^+$  and a vertex  $v^- \in \mathcal{V}^-$ .

We are in position to define a colored tensor graph.

**DEFINITION 30** (Colored tensor graph [10, 13]). A rank  $D$  colored tensor graph  $\mathcal{G}$  is a graph such that:

- $\mathcal{G}$  is  $(D+1)$  colored and bipartite;
- $\mathcal{G}$  is a rank  $D$  tensor graph.

The type of restriction introduced in Definition 30 allows us to have a control on the type of graphs which are generated. This restriction is therefore useful to discuss a specific class of these graphs but still containing a large (infinite) number of them.

As an illustration, a rank 3 colored tensor graph is pictured in Figure 19. Each vertex (looking like the  $K_4$  complete graph again) is the dual of a tetrahedron and an edge represents a triangle endowed with a color  $i \in \{0, 1, 2, 3\}$ . The graph is also bi-partite.

We denote a rank  $D$  colored tensor graph as  $\mathcal{G}(\mathcal{V}, \mathcal{E})$  where, as usual,  $\mathcal{V}$  is its set of vertices and  $\mathcal{E}$  is its set of edges. In a rank  $D$  colored tensor graph, to each pre-flag could be assigned a color so that an edge can only connect pre-flags of the same color. In such a situation, edges will inherit pre-flag colors. Equivalently, one could infer a color for each pre-flag as the color of the edge touching this pre-flag. There are certainly more data worthwhile to be discussed in such a graph.

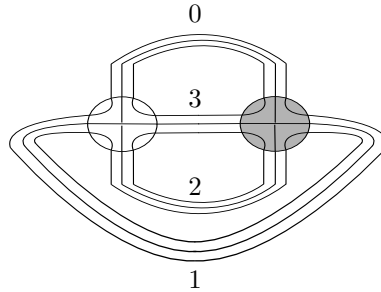


FIGURE 19. A rank 3 colored tensor graph.

**DEFINITION 31** ( $p$ -bubbles [10]). A  $p$ -bubble with colors  $i_1 < i_2 < \dots < i_p$ ,  $p \leq D$ , and  $i_k \in \{0, \dots, D\}$  of a rank  $D$  colored tensor graph  $\mathcal{G}$  is a maximally connected component made of strands of edges of colors  $\{i_1, \dots, i_p\}$ .



For any  $D$ , vertices are 0-bubbles, edges are 1-bubbles. Restricting to  $D = 3$ , there are two other types of  $p$ -bubbles that we shall need in the following:

- The connected components of  $\mathcal{G}$  made with two colors are cycles of lines along which the colors alternate. These are 2-bubbles called the faces of the graph (See the face  $f_{03}$  (in red) in Figure 20).
- A 3-bubble (or simply bubble in  $D = 3$ ) is a connected component of the graph which has three colors (See Figure 20).

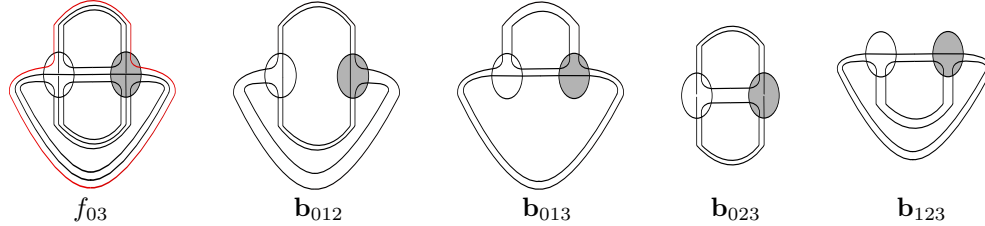


FIGURE 20. The face  $f_{03}$  (in red) and bubbles of the graph of Figure 19.

Faces of stranded graphs are made with strands of edges and vertices. We say that “a face passes through an edge (or vertex)  $x$  time(s)” if it is built with  $x \geq 0$  strands of this edge (or vertex). Of course,  $x = 0$ , means that the face never passes through the element.

Remark that, importantly, having bicolored faces prevents faces to pass more than once in an edge. As in the definition of a face, any strand of the rank  $D$  colored tensor graph is also bicolored. This crucial feature will be used several times in the following.

Any 3-bubble can be seen as a  $2D$  ribbon graph. The vertices of a bubble are three valent vertices obtained by decomposing the vertex of the graph in the way of Figure 20. The edges of a 3-bubble are colored ribbon edges generated by the decomposition of the colored edges of the rank  $D$  colored graph. As ordinary ribbon graphs, 3-bubbles have faces as well. These faces are endowed with a pair of colors like the faces of the initial graph. Thus a bubble is simply a rank 2 colored tensor graph lying inside the rank 3 colored tensor graph. The set of 3-bubbles will be denoted by  $\mathcal{B}_3$  and  $|\mathcal{B}_3| = B_3$ . The index 3 will be omitted when discussing  $D = 3$ .

In the dual picture, a face is a (boundary) surface and a bubble is a collection of surfaces which close and form a  $3D$  region in the simplex. Often, one associates directly the  $3D$  region to the bubble.

A rank  $D$  colored tensor graph admits another equivalent representation. By collapsing the entire rank  $D$  colored tensor graph in the way previously mentioned, namely by forgetting its stranded structure, we obtain another representation of the same graph. This representation, called *compact* in the following, is merely obtained by regarding the tensor graph as a simple bipartite colored graph in the sense of Definition 29. It is worth underlining that one representation of the graph determines unambiguously the other (this is not the case for the collapsed graph which cannot fully characterize the stranded structure of its generating graph). For instance, the graph of Figure 19 can be also drawn in the compact form of Figure 21.

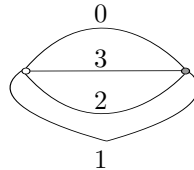


FIGURE 21. Compact representation of the graph of Figure 19.

Rank  $D$  flags can be considered as well on rank  $D$  colored tensor graph, provided these flags possess a color and their gluing respects the graph coloring at each vertex. Thus to axioms (i') and (ii') of Definition 23, we add:

(iii') to each edge and flag, one assigns a color  $i \in \{0, 1, \dots, D\}$ .

The cut of an edge can be done in the same sense of Definition 26 for colored tensor graphs. The crucial point is solely to respect the color structure of the graph after the cut such that each of resulting flags possess the same color structure of the former edge.

The definitions of a rank  $D$  colored tensor graph with flags  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  (see Figure 22 for an illustration for  $D = 3$ ) and its spanning c-subgraphs of  $\mathcal{G}$  follow from spanning c-subgraphs of the stranded graph  $\mathcal{G}$  and from Definition 10. The only point to be added is the color structure. The following is straightforward:

**PROPOSITION 8.** *(Spanning) c-subgraphs of a rank  $D$  colored tensor graph are rank  $D$  colored tensor graphs.*

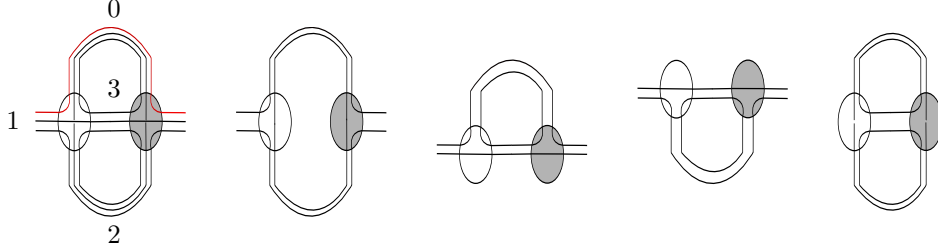


FIGURE 22. An open rank 3 colored tensor graph and its bubbles;  $f_{01}$  (highlighted in red) is an open face;  $\mathbf{b}_{012}$ ,  $\mathbf{b}_{013}$  and  $\mathbf{b}_{123}$  are open bubbles and  $\mathbf{b}_{023}$  is closed.

Let us come back to the elements of the graphs. An edge  $e$  is called bridge if  $\mathcal{G} \vee e$  becomes disconnected. There is no self-loop at this moment.

As in the case of ribbon graphs with flags, introducing tensor graphs with flags requires a special treatment for faces. Using Definition 19, faces can be open or closed connected components if they pass through external points or not. An open face with color pair (01) is highlighted in red in Figure 22. The sets of closed and open faces will be denoted by  $\mathcal{F}_{\text{int}}$  and  $\mathcal{F}_{\text{ext}}$ , respectively. Hence, for a rank  $D$  colored graph with flags, the set  $\mathcal{F}$  of faces is the disjoint union  $\mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$ . The notion of closed or open rank 3 colored tensor graph can be reported accordingly if  $\mathcal{F}_{\text{ext}} = \emptyset$  or not, respectively.

A bubble is open or external if it contains open faces otherwise it is closed or internal. Open and closed bubbles for a rank 3 colored tensor graph with flags given in Figure 22 have been illustrated therein. The sets of closed and open bubbles will be denoted by  $\mathcal{B}_{\text{int}}$  and  $\mathcal{B}_{\text{ext}}$ , respectively.

The notions of pinched graph  $\tilde{\mathcal{G}}$  and of boundary graph is of great significance and we can address the pinching procedure colored tensor graph with flags. First, pinching a tensor flag can be defined by the identification of the external points of the flag. The pinched graph  $\tilde{\mathcal{G}}(\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{\mathfrak{f}}^0)$  is a closed graph obtained from  $\mathcal{G}$  after pinching of all of its flags and still the following relations are valid

$$\tilde{\mathcal{V}} = \mathcal{V}, \quad \tilde{\mathcal{E}} = \mathcal{E}, \quad |\tilde{\mathfrak{f}}^0| = |\mathfrak{f}^0|. \quad (57)$$

**DEFINITION 32** (Boundary tensor graph). • *The boundary graph  $\partial\mathcal{G}(\mathcal{V}_{\partial}, \mathcal{E}_{\partial})$  of a rank  $D$  colored tensor graph  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  is a graph obtained by inserting a vertex with degree  $D$  at each additional flags of  $\mathcal{G}$  and taking the external faces of  $\mathcal{G}$  as its edges. Thus,  $|\mathcal{V}_{\partial}| = |\mathfrak{f}^0|$  and  $\mathcal{E}_{\partial} = \mathcal{F}_{\text{ext}}$ .*

• *The boundary of a closed rank  $D$  colored tensor graph is empty.*

The boundary of a tensor graph without colors is obtained in the same way as defined above since the colored feature of the graph is not used in the definition. To the graph of Figure 22, we associate its boundary depicted in Figure 23.

The dual of rank 3 colored tensor graph with flags is nothing but a 3D simplicial complex with tetrahedra (as vertices) made with colored triangles having a colored 2D boundary [12]. This

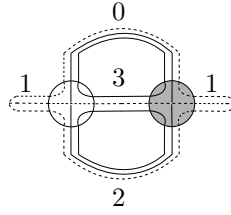


FIGURE 23. Boundary graph (in dashed lines) of the graph of Figure 22.

colored boundary surface is obtained by gluing colored triangles dually associated with colored flags.

Let us stress that the boundary graph  $\partial\mathcal{G}$  of a rank  $D$  (colored or not) tensor graph  $\mathcal{G}$  encodes, in the dual picture, the triangulation of the boundary of the  $D$  simplicial complex dual to  $\mathcal{G}$ . In the specific instance of 3D,  $\partial\mathcal{G}$  represents a 2D triangulation of the boundary of a 3-simplicial complex dual to  $\mathcal{G}$ . In the same vein that colors improve the topology of the triangulated object associated with  $\mathcal{G}$ , a colored boundary graph  $\partial\mathcal{G}$  improves the regular feature of the boundary manifold for that triangulated object. The construction of boundary graph  $\partial\mathcal{G}$  by Definition 32 should be nothing but a compact representation of a more elaborate graph which can encode the triangulation of a surface. We first need another type of colored structure for graphs in order to discuss this.

**DEFINITION 33 (Ve-colored graphs).** • A  $(D+1)$  ve-colored graph is a graph such that to each vertex is assigned a color in  $\{0, \dots, D\}$  and to each edge is assigned a pair  $(ab)$  of colors,  $a, b$  in  $\{0, \dots, D\}$ ,  $a \neq b$ , such that

- either the edge is incident to vertices with the same color  $c$ , in this case,  $a$  (or  $b$ ) equals  $c$  and  $b \in \{0, \dots, D\} \setminus \{a\}$  (respectively,  $a \in \{0, \dots, D\} \setminus \{b\}$ );
  - the edge is incident to vertices with different colors  $a$  and  $b$
- and such that two adjacent edges cannot share the same couple of colors.

Some 4 ve-colored graphs are drawn in Figure 24.

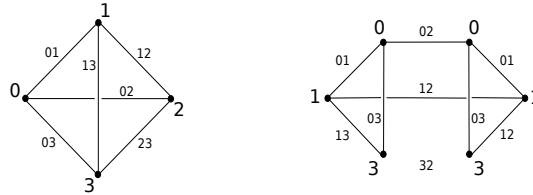


FIGURE 24. Examples of 4 ve-colored graphs.

One notices that a  $(D+1)$  ve-colored graph is a  $D(D+1)/2$  colored graph for edges equipped color pairs  $ab$ ,  $a < b$ , chosen in  $\{01, 02, \dots, 0D, 12, \dots, D(D-1)\}$ . Thus in a ve-colored graph there is no self-loops.

**DEFINITION 34 (Ve-colored tensor graph).** A rank  $D$  ve-colored tensor graph  $\mathcal{G}$  is a graph such that:

- $\mathcal{G}$  is  $(D+2)$  ve-colored;
- $\mathcal{G}$  is a rank  $D$  tensor graph.

The following statement holds.

**PROPOSITION 9 (Color structure of  $\partial\mathcal{G}$  [12]).** The boundary graph  $\partial\mathcal{G}(\mathcal{V}_\partial, \mathcal{E}_\partial)$  of a rank  $D$  colored tensor graph  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathbf{f}^0)$  is a rank  $(D-1)$  ve-colored tensor graph.

**PROOF.** The colors of vertices in  $\partial\mathcal{G}$  inherits the colors of the flags of  $\mathcal{G}$ . The edge coloring should clearly coincide with the external face coloring defined by a couple  $(ab)$ . At one flag, the

color pairs of any external face never coincide. This makes the graph  $\partial\mathcal{G}$  as  $(D + 1)$  ve-colored. To construct the extra tensor structure on  $\partial\mathcal{G}$ , one expands its vertices and edges into stranded vertices with fixed coordination  $D$  and edges with  $D - 1$  parallel strands. The connecting pattern of the pre-flags of its vertices follows the one of the complete graph  $K_D$ . The proposition follows.  $\square$

For instance in 3D, each vertex of  $\partial\mathcal{G}$  is colored and is three valent. Each edge of  $\partial\mathcal{G}$  is a bicolored ribbon. See Figure 25.

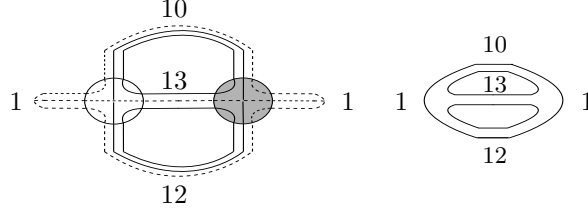


FIGURE 25. Rank 2 ve-colored stranded structure of the boundary graph of Figure 23.

REMARK 1. *Importantly, any face of the boundary graph coincide with a connected component of the boundary  $\partial\mathbf{b}_{aa_1a_2}$  of some open bubble  $\mathbf{b}_{aa_1a_2}$  viewed as an open ribbon graph. Indeed, any (closed) face of a boundary graph is made with strands coming from open faces belonging necessarily to open bubbles. The boundary graph being closed, in order to get all of its faces, one has first to pinch all external bubbles  $\mathbf{b}$ , yielding  $\tilde{\mathbf{b}}$ , and collect closed faces coming uniquely from the pinching as performed in Proposition 4.*

As an illustration of the above remark, consider the graph  $\mathcal{G}$  Figure 22 with boundary  $\partial\mathcal{G}$  drawn in Figure 23. Let us pick the face  $f_{012}$  of  $\partial\mathcal{G}$  and consider the open bubble  $\mathbf{b}_{012}$  and its pinched version  $\tilde{\mathbf{b}}_{012}$ . The closed face  $\tilde{f}_{012}$  of  $\tilde{\mathbf{b}}_{012}$  coincide with  $f_{012}$ . Furthermore, any such closed face cannot belong to two different pinched bubbles by color exclusion and by the fact that there are only two bubbles passing through an edge which could use a given edge strand.

**4.2. W-colored stranded graphs.** It has been discussed in length that the contraction of an edge in a tensor colored graph yields another type of graph for which neither the color (Definition 29) nor the tensor axioms (Definition 23) apply [12]. In order to define the topological polynomial and circumvent such an odd feature of tensor graphs, some proposals have been made to redefine the notion of contraction of an edge or redefine subgraphs for which the contraction applies [12, 18].

Using Definition 27, we can now contract any rank  $D$  edge provided the fact that we are working in the extended framework of stranded graphs. This definition therefore applies to a colored tensor graph. Our task is to characterize the new class of stranded graph obtained after an edge contraction of a colored tensor graph. From now on edge contraction means always soft edge contraction in the sense Definition 27.

The contraction of an edge  $e$  in a rank  $D$  colored tensor graph  $\mathcal{G}$  (possibly with flags) yields a rank  $D$  stranded graph  $\mathcal{G}/e$  obtained from  $\mathcal{G}$  by the procedure described in Definition 27 and respecting the coloring. By respecting the coloring, we understand that edges which are not contracted keep their color and all strands keep their color pair. For a rank  $D$  colored tensor graph, it is straightforward to see that any edge contraction in  $\mathcal{G}$  brings in  $\mathcal{G}/e$  a vertex possessing at least two pre-flags of the same color (see an example in  $D = 3$  in Figure 26). This breaks the color axiom on vertices since it allows that edges with the same color to be branched on the same vertex. But still in  $\mathcal{G}/e$  there is a color structure defined in a sense weaker than Definition 29. Such a colored structure is mainly based on the feature that any edge contraction preserves strands and their color pair.

In the following, we first spell out the object of interest as defined by the result of the contraction of a rank  $D$  colored tensor graph. Then we find a characterization of it.

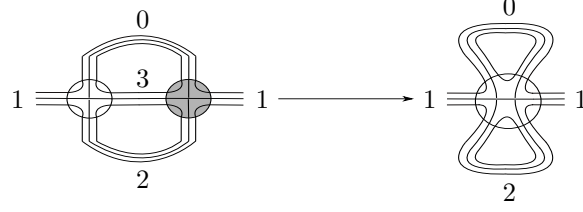


FIGURE 26. Contraction of an edge in a rank 3 colored tensor graph with flags.

DEFINITION 35 (Rank  $D$  w-colored graph I). *A rank  $D$  weakly colored or w-colored graph is the equivalence class (up to trivial discs) of a rank  $D$  stranded graph obtained by successive edge contractions of some rank  $D$  colored tensor graph.*

The equivalence class of any rank  $D$  colored tensor graph up to trivial discs is a rank  $D$  w-colored graph. This holds by convention when no contraction is performed. Another property is that several rank  $D$  colored tensor graphs may have the same contracted graph. Hence to one rank  $D$  w-colored graph may correspond several rank  $D$  colored tensor graphs which could generate it. On the other hand, the existence of a representative  $\mathcal{G}$  of some rank  $D$  w-colored graph  $\mathfrak{G}$  as being the contraction of some rank  $D$  colored tensor graph does not imply necessarily that any other representative results from a graph contraction. Given  $\mathcal{G}$  the representative of  $\mathfrak{G}$  and  $\mathcal{G}_{\text{color}}$  the rank  $D$  colored tensor graph which gives  $\mathcal{G}$  after contraction, let us assume that this contraction yields  $\ell_0$  extra discs with particular bi-colored strands. Any representative  $\mathcal{G}'$  of  $\mathfrak{G}$ ,  $\mathcal{G}'$  could come from the contraction of a rank  $D$  colored tensor graph  $\mathcal{G}'_{\text{color}} \sim \mathcal{G}_{\text{color}}$  if the number  $k$  of discs of  $\mathcal{G}'$  is larger than  $\ell_0$ , in which case  $\mathcal{G}'_{\text{color}} = (\mathcal{G}_{\text{color}} \setminus D_{\mathcal{G}_{\text{color}}}) \cup D_{k-\ell_0}$ , where  $D_{k-\ell_0} = D_{\mathcal{G}'} \setminus D_{\ell_0}$  and where  $D_{\ell_0}$  is the set of additional discs coming uniquely from the contraction of  $\mathcal{G}_{\text{color}}/D_{\mathcal{G}_{\text{color}}}$ . One should pay attention of the type of color pairs that should occur on discs of  $\mathcal{G}'$  to match with the color pairs of discs obtained after contractions.

It is relevant to discuss in greater detail the type of (non trivial) vertices that will generate any representative of any rank  $D$  w-colored graphs. To address this, we observe the following result from Lemma 3 in [4] and slightly adapted to the present context.

LEMMA 1 (Full contraction of a tensor graph). • *Contracting an edge in a rank  $D$  (colored) tensor graph does not change its boundary.* • *The contraction of all edges in arbitrary order of a rank  $D$  (colored) tensor graph  $\mathcal{G}$  yields a graph  $\mathcal{G}^0$  defined by the boundary  $\partial\mathcal{G}$  up to additional discs.*

PROOF. Contracting one edge in  $\mathcal{G}$ , all flags and all external faces remain untouched, and no other external face can be created by such a move (we remove all present inner faces and all remaining faces get shorter but are never cut). Hence for the resulting stranded graph  $\mathcal{G}/e$ , one has  $\partial(\mathcal{G}/e) = \partial\mathcal{G}$ . By iteration, contracting all edges of  $\mathcal{G}$  in arbitrary order, the resulting graph  $\mathcal{G}^0$  should contain  $\partial\mathcal{G}$  as its boundary. But contracting all edges of  $\mathcal{G}$ , one obtains a single vertex graph per connected component plus trivial discs and no other closed faced left otherwise it would mean that there exist still edges where these closed faces pass. Hence necessarily  $\mathcal{G}^0$  is nothing but a graph made only with vertices (without edges) defined therefore by  $\partial\mathcal{G}^0 = \partial\mathcal{G}$  up to trivial discs.  $\square$

Let  $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  be a connected rank  $D$  colored tensor graph with a nonempty set of flags,  $\mathfrak{f}^0 \neq \emptyset$ . After a full contraction of all edges of  $\mathcal{G}$ , Lemma 1 tells us that the end result is totally encoded in the boundary graph  $\partial\mathcal{G}$  up to some discs. This graph  $\mathcal{G}^0$  appears as one or a collection of rank  $D$  stranded vertices  $v_{\partial\mathcal{G};k}$ ,  $k$  parametrizing the number of connected components of  $\partial\mathcal{G}$ . Note that if  $\mathfrak{f}^0 = \emptyset$  then there is no boundary in  $\mathcal{G}$  and  $v_{\partial\mathcal{G}}$  is empty.

Consider a stranded graph made with such vertices  $v_{\partial\mathcal{G}_i; k_i}$  for a finite family of boundaries  $\partial\mathcal{G}_i$  of some connected rank  $D$  colored tensor graphs  $\mathcal{G}_i$  with a nonempty set of flags,  $i \in I$ ,  $I$  finite. In order to glue these vertices, one needs edges of rank  $D$ .

PROPOSITION 10 (Rank  $D$  w-colored stranded graphs II). *A rank  $D$  stranded graph  $\mathcal{G}$  is a representative of a rank  $D$  w-colored graph  $\mathfrak{G}$  if and only if it satisfies*

- (i'') *vertices  $v_i$  of  $\mathcal{G}$  attached with flags are graphs  $V_i$  coming from the boundaries  $\partial\mathcal{G}_i$  of a family of connected rank  $D$  colored tensor graph  $\mathcal{G}_i(\mathcal{V}_i, \mathcal{E}_i, \mathfrak{f}_i^0)$  with  $\mathfrak{f}_i^0 \neq \emptyset$ . All pre-flags of  $\mathcal{G}$  have a fixed cardinal  $D$ ;*
- (ii') *the edges of  $\mathcal{G}$  are of rank  $D$  and with parallel strands;*
- (iii') *to edges and flags of  $\mathcal{G}$  are assigned a color  $\in \{0, 1, \dots, D\}$  inherited from the pre-flags of the  $v_i$ 's.*

PROOF. Let us consider the representative  $\mathcal{G}$  of a rank  $D$  w-colored graph  $\mathfrak{G}$  and  $\mathcal{G}_{\text{color}}$  a rank  $D$  colored tensor graph the contraction of which yields  $\mathcal{G}$ . Let us pick another representative  $\mathcal{G}'$  of  $\mathfrak{G}$ . Properties (ii') and (iii') for edges and flags of  $\mathcal{G}'$  are trivially satisfied.

Next, observe that to any rank  $D$  colored tensor graph corresponds a rank  $D$  w-colored tensor graph which is the class of this same graph. Hence if  $\mathcal{G}$  contains the vertices  $v_0$  and  $\bar{v}_0$  (from bi-partite property) of ordinary colored tensor graphs, we can define these two vertices as the boundaries of the graphs defined by themselves attached with flags; the cardinal of their pre-flags is fixed to  $D$  by definition. It is obvious that if  $v_0$  and  $\bar{v}_0$  (which are not trivial discs) are in  $\mathcal{G}$  they are also in  $\mathcal{G}'$  and so the same construction applies to  $\mathcal{G}'$ .

Any vertex  $v$  in  $\mathcal{G}$  which is not a disc and not of the form of  $v_0$  and  $\bar{v}_0$  comes necessarily from some contractions of some subset  $\{e_l\}_{l \in I_v}$  of edges in  $\mathcal{G}_{\text{color}}$ . Note that for all  $v$ ,  $\{e_l\}_{l \in I_v}$  should define a partition of edges of  $\mathcal{G}_{\text{color}}$  which do not belong to  $\mathcal{G}$ . Consider the c-subgraph  $\mathcal{G}_v$  in  $\mathcal{G}_{\text{color}}$  defined by  $\{e_l\}_{l \in I_v}$  and their end vertices. By Proposition 8,  $\mathcal{G}_v$  is a rank  $D$  colored tensor graph. Then, after the full contraction of all the  $e_l$ 's, by Lemma 1, we obtain  $\partial\mathcal{G}_v$  up to some discs.  $\partial\mathcal{G}_v$  also coincides with the new vertex  $v$  in  $\mathcal{G}$  attached with flags. Proceeding by iteration on all vertices of the representative  $\mathcal{G}$ , we get a family of rank  $D$  colored c-subgraphs  $\mathcal{G}_v \subset \mathcal{G}_{\text{color}}$ , the overall contraction of which yields the vertices  $v$  (plus flags) of  $\mathcal{G}$  up to some trivial discs. The cardinal of all present pre-flags is obviously fixed to  $D$ . Any vertex  $v$  in  $\mathcal{G}$  which is not a disc is also present in  $\mathcal{G}'$  therefore all vertices of  $\mathcal{G}'$  come from boundaries of some families of colored graphs.

Now consider a rank  $D$  stranded graph  $\mathcal{G}$  satisfying (i''), (ii') and (iii') and its equivalence class  $\mathfrak{G}$ . Then by replacing in  $\mathcal{G}$  all the vertices  $v_i$  by the colored graphs  $\mathcal{G}_i$  and respecting the coloring, we obtain a rank  $D$  colored tensor graph  $\mathcal{G}_{\text{color}}$ . Contracting in  $\mathcal{G}_{\text{color}}$  all edges in all  $\mathcal{G}_i$ 's yields again the  $v_i$ 's determined by their boundary (Lemma 1), up to some discs. Thus the end result  $\mathcal{G}'$  of these edge contractions is  $\mathcal{G}$  up to some discs therefore belongs to  $\mathfrak{G}$ . Hence  $\mathfrak{G}$  is a rank  $D$  w-colored graph. □

From now on, since the rank of edges, flags and pre-flags is fixed to  $D$ , we omit to mention it. Thus, when no confusion may occur, rank  $D$  colored tensor graph are named colored graphs and rank  $D$  w-colored graphs are named w-colored graphs.

The key notion of interest here pertains again to the spanning c-subgraphs for a w-colored graph. This notion has been already mentioned for equivalent classes, the sole new point here is to add colors.

PROPOSITION 11 (c-subgraph). *Let  $\mathfrak{G}$  be a rank  $D$  w-colored graph. The c-subgraphs of  $\mathfrak{G}$  admit a rank  $D$  w-colored graph structure.*

PROOF. Consider  $\mathfrak{G}$  a w-colored graph and  $\mathcal{G}$  the representative in  $\mathfrak{G}$  obtained after contraction of some colored graph  $\mathcal{G}_{\text{color}}$ . Let  $\mathfrak{g} \subset \mathfrak{G}$  be a c-subgraph of  $\mathfrak{G}$ . For any such  $\mathfrak{g}$ , we shall show that there exists a subgraph  $g$  of  $\mathcal{G}$  such that  $g$  is a representative of  $\mathfrak{g}$ . Then the proof will be complete because  $g$  is exactly the colored graph that we seek since it comes from a contraction of some subgraph of  $\mathcal{G}_{\text{color}}$  admitting itself a colored structure by Proposition 8. Now, the existence of  $g$  is guaranteed by the following. Choose any representative  $g'$  of  $\mathfrak{g}$  in any  $\mathcal{G}'$ , then  $g'$  is a c-subgraph in  $\mathcal{G}'$ . From  $g'$ , we built  $\tilde{g} = g' \setminus D_{g'}$  which is a c-subgraph of  $\mathcal{G}$ . Expanding  $\tilde{g} \subset \mathcal{G}$  back in  $\mathcal{G}_{\text{color}}$ , we get a colored c-subgraph of  $\mathcal{G}_{\text{color}}$  the contraction of which yields finally a c-subgraph  $g$  of  $\mathcal{G}$ . To see that  $g \sim g'$  is direct since  $g \sim \tilde{g}$ .

□

We give now few more properties of the vertices of representatives of rank  $D$  w-colored graphs. First, there are infinitely many vertices for these graphs. However, some constraints should be imposed on vertices for being the boundaries of some rank  $D$  colored tensor graphs.

LEMMA 2. *Consider a rank  $D$  w-colored graph with representative made with a vertex with  $N$  pre-flags. If  $D$  is odd then  $N$  should be always even.*

PROOF. This simply holds from the fact that the number  $ND$  of pre-flag points in the vertex considered of the representative should be even in order to properly form bows.

□

For rank  $D$  tensor graphs, the above lemma is obvious since  $N$  is always fixed to  $D - 1$ . The point of the above proposition is to establish that a vertex of rank  $D$  w-colored graph can be of arbitrary degree but with even parity in odd dimension.

The next lemma follows arguments given in Lemma 2.1 in [3] (as a dictionary, external legs in quantum field theory corresponds to flags). For sake of completeness, let us restate this proposition in the present language.

LEMMA 3 (Lemma 2.1 in [3]). *Given a rank  $D$  colored tensor graph with  $n$  vertices and  $N$  flags. If one color is missing on its flags, then*

- $n$  is an even number
- the number  $N$  of flags is even and the colors of flags should appear in pair.

PROOF. If one color, say  $a$ , is missing on the flags, it means that  $a$  appear only on edges. Each edge occupies two pre-flags of the same color and that one has  $n$  pre-flags of each color. Now since  $a$  should only appear on edges, it follows that the  $n$  pre-flags of color  $a$  should be all paired to give edges. This proves the first point.

We henceforth know that  $n$  is even. Assuming that there is an odd number of flags possessing a particular color, this implies that the number of pre-flags with that color which should be paired for making edges would be odd as well, hence would be impossible. The second point follows.

□

Lemma 3 entails other properties of the vertices of representatives of a w-colored graph. Indeed, we infer from this statement that the vertices in a rank  $D$  w-colored graph can be only of some types. Some vertices attached with flags have been depicted in Figure 27 using the compact representation. Note that, in contrast with colored graphs, the compact notation becomes ambiguous for general w-colored graphs. We use it here for displaying the color feature of the different flags attached to the vertices. For convenience, we also introduce the following notation: each pre-flag  $f$  may have a color  $a$  or  $\bar{a}$  which is reminiscent of the fact that  $a$  and  $\bar{a}$  come from different types of vertices in some bipartite graph. We call  $a$  and  $\bar{a}$  **conjugate** colors and call conjugate flags, the flags which possess  $a$  and  $\bar{a}$  as colors.

When a vertex has uniquely one pair of conjugate colors  $a$  and  $\bar{a}$  for all pre-flags, it is related to a so-called tensor unitary invariant object [14]. We call it a 1-color vertex. When the vertex has 2 conjugate color pairs,  $2 < D$ , then it can be seen as a gluing of two 1-color vertices  $v_1$  and  $v_2$  after cutting some bows in  $v_1$  and  $v_2$  and then gluing these (still respecting the coloring). Then one may figure out that the trace invariant object above gets a weaker sense because the invariance is broken in some sector (induced by the cut bows). For more colors, one proceeds by iteration gluing two 1-color vertices and then gluing the result with another 1-color vertex with a different color. Performing such a gluing is possible up to a number of  $D - 1$  colors. Though beyond the scope of this work, an interesting task would be to count the number of such  $2, 3, \dots, D - 1$  w-colored vertices for a fixed number  $N$  of flags. For the last type of vertices with all  $D$  colors, it does not necessarily follow from the gluing of 1-color vertices. The same counting will become highly non trivial, if exists.

LEMMA 4. *Let  $e$  be a regular edge or a bridge with end vertices  $v_1$  and  $v_2$  of any representative of a rank 3 w-colored graph. Contracting  $e$  in the representative yields a new vertex  $v$  which is connected.*

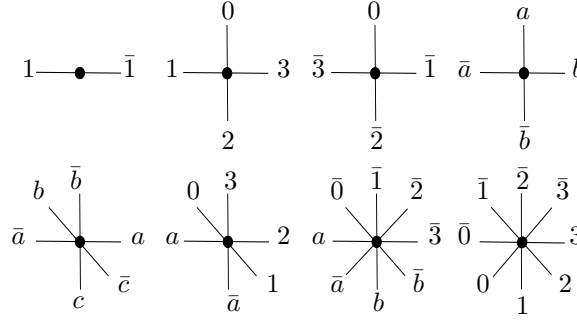
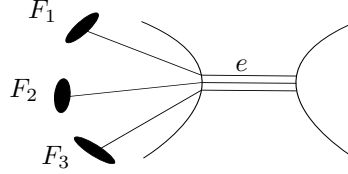


FIGURE 27. Some vertices of w-colored graphs.

PROOF. Consider  $e$  which is not a self-loop,  $e$  is common to all  $\mathcal{G}$  representative of a rank 3 w-colored graph. Its strands join in  $v_1$  and in  $v_2$  a collection of connected families of pre-flags. The initial vertices  $v_1$  and  $v_2$  being connected, two different cases may occur:

(i) All strands of  $e$  are connected to a unique pre-flag either in  $v_1$  or  $v_2$ . This implies that there are just two pre-flags in  $v_1$  or in  $v_2$ . Contracting  $e$ , the vertex  $v$  obtained is connected.

(ii) Assume now that attached to each strand of  $e$ , we have  $d$  families of connected pre-flags in  $v_1$  or in  $v_2$ ,  $2 \leq d \leq 3$  (see an illustration in Figure 28). This is enforced by the coloring which prevents faces to pass through  $e$  more than once. If contracting  $e$  gives a disconnected vertex then, in each  $v_1$  and  $v_2$ , the connected families of pre-flags do not have any bow between them<sup>7</sup>. Notice that there is necessarily at least one connected family of pre-flags joining  $e$  by a unique bow  $b$ . For such a family  $\xi$ , all pre-flags are fully connected via bows and there is a unique pre-flag point, the one related to  $e$  via  $b$ , not linked to any other pre-flag in  $\xi$ . By an argument of parity, we aim at showing that it is impossible.

FIGURE 28. Families  $F_1, F_2$  and  $F_3$  of connected pre-flags attached to each strand of a non self-loop edge.

For simplicity, let us fix that  $e$  is of color 0 and that the strand  $s$  of color pair  $(01)$  in  $e$  touches the connected and separate family  $\xi$  a pre-flag  $f$ . Two cases may occur: either  $f$  is of color 1 or of color  $\bar{0}$ .

(A) If  $f$  of color 1: Denote  $N_i$  and  $\bar{N}_i$  the number of pre-flags in  $\xi$  of color  $i = 0, 1, 2, 3$  and  $\bar{i} = \bar{0}, \bar{1}, \bar{2}, \bar{3}$ , respectively. There are exactly  $N_i + \bar{N}_i$  pre-flag points labeled by color pairs  $(ij)$ ,  $i = 0, 2, 3, j \in \{0, 1, 2, 3\}$  but  $j \neq i$ . These points come from pre-flags of colors 0, 2, 3 and  $\bar{0}, \bar{2}, \bar{3}$ . Meanwhile, there are  $(N_1 - 1) + \bar{N}_1$  pre-flag points of color pairs  $(01)$ , and  $N_1 + \bar{N}_1$  of color pairs  $(1j)$ ,  $j \in 2, 3$ . These last points come uniquely from pre-flags with color 1 and  $\bar{1}$ . Assuming that all pre-flag points but one in  $\xi$  should be totally connected by bows implies the following equations:

$$\begin{aligned} N_0 + \bar{N}_0 + (N_1 - 1) + \bar{N}_1 &= 2k^{01}, \\ N_0 + \bar{N}_0 + N_2 + \bar{N}_2 &= 2k^{02}, \\ N_0 + \bar{N}_0 + N_3 + \bar{N}_3 &= 2k^{03}, \\ N_1 + \bar{N}_1 + N_2 + \bar{N}_2 &= 2k^{12}, \\ N_1 + \bar{N}_1 + N_3 + \bar{N}_3 &= 2k^{13}, \\ N_2 + \bar{N}_2 + N_3 + \bar{N}_3 &= 2k^{23}, \end{aligned} \tag{58}$$

<sup>7</sup>Note that this is only a necessary condition for obtaining a disconnected vertex.



for some  $k^{ij}$  positive integers counting the number of bow with colors  $(ij)$ . It is simple to show that this system leads to the inconsistent equation

$$2(N_0 + \bar{N}_0) + 2(k^{12} - k^{02}) - 1 = 2k^{01}. \quad (59)$$

(B) If  $f$  is of color  $\bar{0}$ :  $N_0 + (\bar{N}_0 - 1)$  becomes the number of pre-flag points of color pairs  $(01)$  in  $\xi$  induced by pre-flags of color 0 and  $\bar{0}$  and  $N_1 + \bar{N}_1$  is the number of pre-flags points of colors  $01$  induced by pre-flags of color 1 and  $\bar{1}$ . This is the same as switching the role of 1 and  $\bar{0}$  in the above case. The above system holds again as constraints on the number of pre-flag points and yields the same conclusion.  $\square$

The above lemma should clearly admit an extension to any dimension  $D$  by partitioning  $D$  and analyzing the connection of the strands of  $e$  and the connected pre-flag families. We do not need however such a stronger and much technical result for the following.

In contrast, the contraction of a self-loop may disconnect the vertex. In any case, since self-loop graphs are terminal forms, a special treatment for them is required.

The whole construction above is mainly made in order to maintain the consistency of the  $p$ -bubble definition. For instance, edges are still 1-colored objects, (closed and open) faces are bicolored connected objects, (closed and open) 3-bubbles are connected objects with 3 colors. However, 3-bubble are no longer built uniquely with three valent vertices. They can be made with vertices with lower or greater valence as illustrated in Figure 29. In any case, Definition 31 is still valid and we shall focus on this.

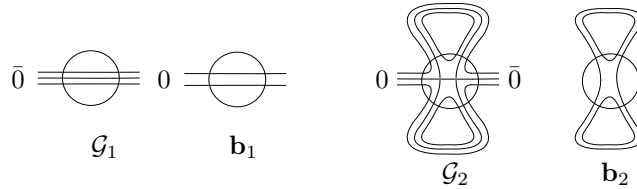


FIGURE 29. Two bubbles graphs  $\mathbf{b}_1$  and  $\mathbf{b}_2$  of  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , respectively:  $\mathbf{b}_1$  has a vertex with valence 2 and  $\mathbf{b}_2$  a vertex with valence 4.

LEMMA 5. Let  $\mathcal{G}$  be a representative of  $\mathfrak{G}$  a rank  $D$   $w$ -colored graph. • Cutting an edge  $e$  in  $\mathcal{G}$  yields a representative  $\mathcal{G} \vee e$  of another rank  $D$   $w$ -colored graph denoted  $\mathfrak{G} \vee e$ . • Contracting an edge  $e$  in  $\mathcal{G}$  yields a representative  $\mathcal{G}/e$  of another rank  $D$   $w$ -colored graph denoted  $\mathfrak{G}/e$ .

PROOF. Let  $e$  be an edge in  $\mathcal{G}'$  representative of some  $w$ -colored graph  $\mathfrak{G}$  and  $\mathcal{G}$  the representative of  $\mathfrak{G}$  resulting from a contraction of some colored graph  $\mathcal{G}_{\text{color}}$ .

The cut graph  $\mathcal{G}' \vee e$  is equivalent to  $\mathcal{G} \vee e$ . We use again here the expansion and contraction move: starting from  $\mathcal{G} \vee e$ , we expand this graph in  $\mathcal{G}_{\text{color}} \vee e$  and contract it back to nothing but itself. One finds that  $\mathcal{G} \vee e$  is therefore a graph resulting from the contraction of a colored graph. Its equivalence class is the  $w$ -colored graph desired.

Contracting now  $e$  in  $\mathcal{G}'$ , one gets a graph  $\mathcal{G}'/e$  equivalent to  $\mathcal{G}/e$ . There exists a colored graph  $\mathcal{G}_{\text{color}}^0$  contracting to  $\mathcal{G}/e$ . This is nothing but  $\mathcal{G}_{\text{color}}$  on which after all contraction yielding  $\mathcal{G}$ , one perform another contraction on  $e$ . Note that we could have defined  $\mathcal{G}_{\text{color}}^0 = \mathcal{G}_{\text{color}}/e$  and perform the same series of contraction bringing initially  $\mathcal{G}$ . The end result is  $\mathcal{G}/e$ . Thus the equivalence class of  $\mathcal{G}/e$  is the  $w$ -colored graph that we seek.  $\square$

We henceforth restrict the rank of graphs to  $D = 3$  and aim at studying the universal invariant polynomial extending BR polynomial for this new category of graphs. The extension for any  $D$  should require more work but could be almost inferred from the subsequent analysis. From now rank 3  $w$ -colored graphs will be called more simply  $w$ -colored graphs.

Mostly, we have studied non self-loop edges so far and their main properties under contraction and cut can be guessed in any dimension. Self-loops are more subtle and their behavior under contraction is much more involved. Restricting to dimension 3 will simplify the analysis.

Self-loops find also a meaning in the present situation. A self-loop can be at most a 3-inner edge. A 3-inner edge determines by itself a graph, see Figure 30O. If it is a 2-inner self-loop then  $e$  should be of the form illustrated in Figure 30A. Otherwise,  $e$  may be a 1-inner self-loop (Figure 30B) or a 0-inner (Figure 30C).

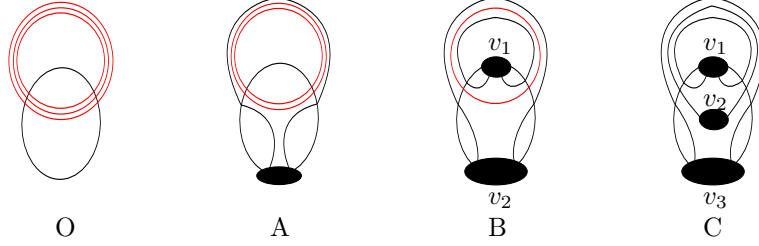


FIGURE 30.  $p$ -inner self-loops: a 3-inner (O) and a 2-inner (A) self-loop with their unique possible configuration; a 1-inner self-loop with possible two sectors  $v_1$  and  $v_2$  (B) and a 0-inner self-loop with three possible sectors  $v_1, v_2$  and  $v_3$  (C).

For a 2-inner self-loop  $e$ , the outer strand of  $e$  should be in contact with other flags or edges, otherwise  $e$  would be a 3-inner edge. We represent the rest of the vertex with possible edges and flags attached by a black diagram in Figure 30A.

A 1-inner self-loop  $e$  has two outer strands which should be in contact with two sectors of the vertex,  $v_1$  and  $v_2$  as illustrated in Figure 30B, such that  $v_1$  and  $v_2$  have at least two flags or edges. Remark that nothing prevents  $v_1$  and  $v_2$  to have pre-flags in common or to be linked by bows. In this situation, we shall say that these sectors coincide otherwise we call them separate. A 2-inner self-loop  $e$  has three such sectors,  $v_1, v_2$  and  $v_3$ , which can pairwise coincide or can be pairwise separate. Separate sectors should contain each at least two flags or edges.

The notion of trivial self-loop can be addressed at this point for this category of graphs. We call a self-loop  $e$  trivial if it is a 3-inner or 2-inner self-loop or if it is a 1-inner or 0-inner self-loop such that there is no edge between separate sectors  $v_i$ .

For a 3-inner self-loop, the contraction gives three trivial discs, see Figure 31O. For a trivial 2-inner self-loop the contraction is still straightforward and yields Figure 31A. For a 1-inner self loop contraction (see Figure 31B), one gets one extra disc and has two possible configurations: either the vertex remains connected or it gets disconnected with two (possibly non trivial) vertices in both situations. Concerning the contraction of a trivial 1-inner self-loop, it is immediate that the vertex gets disconnected in two non trivial vertices. For a 0-inner self-loop contraction (see Figure 31C), we have no additional disc but three types of configurations with up to three disconnected (and possibly non trivial) vertices. The contraction of a trivial 0-inner self-loop yields directly three disconnected and non trivial vertices.

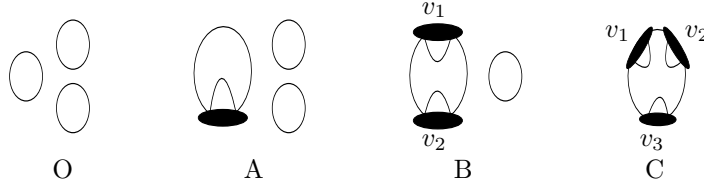


FIGURE 31.  $p$ -inner self-loop contractions corresponding to O, A, B and C of Figure 30, respectively.

**LEMMA 6** (Rank 3 trivial self-loop contraction). *Let  $\mathfrak{G}$  a  $w$ -colored graph and  $\mathcal{G}$  any of its representative with boundary  $\partial\mathcal{G}$ ,  $e$  be a trivial self-loop of  $\mathcal{G}$ , and  $\mathcal{G}' = \mathcal{G}/e$  the result of the contraction of  $\mathcal{G}$  by  $e$  with boundary denoted by  $\partial\mathcal{G}'$ . Let  $k$  denote the number of connected components of  $\mathcal{G}$ ,  $V$  its number of vertices,  $E$  its number of edges,  $F_{\text{int}}$  its number of faces,  $B_{\text{int}}$  its number of closed bubbles and  $B_{\text{ext}}$  its number of open bubbles,  $C_{\partial}$  the number of connected*

component of  $\partial\mathcal{G}$ ,  $f = V_\partial$  its number of flags,  $E_\partial = F_{\text{ext}}$  the number of edges and  $F_\partial$  the number of faces of  $\partial\mathcal{G}$ , and let  $k', V', E', F_{\text{int}}, C'_\partial, B'_{\text{int}}, B'_{\text{ext}}, C'_\partial, f', E'_\partial$ , and  $F'_\partial$  denote the similar numbers for  $\mathcal{G}'$  and its boundary  $\partial\mathcal{G}'$ .

- If  $e$  is a 3-inner self-loop, then

$$\begin{aligned} k' &= k + 2, & V' &= V + 2, & E' &= E - 1, & F'_{\text{int}} &= F_{\text{int}}, \\ C'_\partial &= C_\partial, & f' &= f, & E'_\partial &= E_\partial, & F'_\partial &= F_\partial, \\ B'_{\text{int}} &= B_{\text{int}} - 3, & B'_{\text{ext}} &= B_{\text{ext}}. \end{aligned} \quad (60)$$

- If  $e$  is a 2-inner self-loop, then

$$\begin{aligned} k' &= k + 2, & V' &= V + 2, & E' &= E - 1, & F'_{\text{int}} &= F_{\text{int}}, \\ C'_\partial &= C_\partial, & f' &= f, & E'_\partial &= E_\partial, & F'_\partial &= F_\partial, \\ B'_{\text{int}} &= B_{\text{int}} - 1, & B'_{\text{ext}} &= B_{\text{ext}}. \end{aligned} \quad (61)$$

- If  $e$  is a trivial  $p$ -inner self-loop such that  $p = 0, 1$ , then

$$\begin{aligned} k' &= k + 2, & V' &= V + 2, & E' &= E - 1, & F'_{\text{int}} &= F_{\text{int}}, \\ C'_\partial &= C_\partial, & f' &= f, & E'_\partial &= E_\partial, & F'_\partial &= F_\partial, \\ B'_{\text{int}} + B'_{\text{ext}} &= B_{\text{int}} + B_{\text{ext}} + \alpha_p, \end{aligned} \quad (62)$$

where  $\alpha_p = 3 - 2p$ .

PROOF. • The first situation of a 3-inner self-loop defines a particular closed graph made of a single vertex with a unique self-loop. The contraction destroys the vertex and the three closed bubbles and generate three discs. The result follows.

For  $p$ -inner self loops,  $p \leq 2$ , there should be other flags or edges on the same end vertex. The first lines of (61)-(62) should not cause any difficulty using the definition of the contraction which preserve (open and closed) strands. Let us focus on the number of bubbles of the graph.

• Consider that  $e$  is 2-inner. One first should notice that there is a bubble generated by the two inner faces of  $e$  and that contracting  $e$  destroys this closed bubble. Besides, the other (open or closed) bubbles which passed through  $e$  and which were defined by one of the two inner faces are simply deformed. They loose one face (the inner) but remain in the contracted graph. The second line of (61) follows.

Let us assume now that  $e$  is trivial and 1- or 0-inner with color  $a$ . Consider in the initial vertex  $v$ , the two or three separate sectors of pre-flags  $v_1$ ,  $v_2$  and  $v_3$  determined by the strands of  $e$ . We remind that  $v_1$ ,  $v_2$  and  $v_3$  should contain each at least two flags or edges. Consider as well the three bubbles labeled by colors  $(aa_1a_2)$ ,  $(aa_1a_3)$  and  $(aa_2a_3)$  passing through  $e$ .

• If  $e$  is 1-inner, let us assume that the face  $(aa_3)$  is the one which is inner, without loss of generality. The edge  $e$  decomposes in three ribbon edges each determined by a couple of pairs  $(aa_i; aa_j)$ ,  $i < j$ , such that the strand  $(aa_1)$  has an end point in the sector  $v_1$  and  $(aa_2)$  an end point in  $v_2$ . If  $e$  is trivial and 1-inner, its contraction yields two connected vertices plus one disc. Note that only the bubbles  $\mathbf{b}_{aa_1a_3}$  and  $\mathbf{b}_{aa_2a_3}$  (in the whole graph) include the face  $(aa_3)$ . Contracting  $e$ , the bubbles  $\mathbf{b}_{aa_1a_3}$  and  $\mathbf{b}_{aa_2a_3}$  just loose that face but are still present since in  $v_1$  and  $v_2$ , there exist still faces where  $\mathbf{b}_{aa_1a_3}$  and  $\mathbf{b}_{aa_2a_3}$  passed before. Now the last bubble  $\mathbf{b}_{aa_1a_2}$  gets split in two parts: these parts are disjoint bubbles with the same triple colors since there is no edge in the graph relating them. These bubbles can be open or closed depending on the nature of  $\mathbf{b}_{aa_1a_3}$  in  $v_1$  and  $v_2$ . Hence,  $B'_{\text{int}} + B'_{\text{ext}} = B_{\text{int}} + B_{\text{ext}} + 1$ .

• Now let us discuss the 0-inner self-loop situation with its three separate sectors  $v_1$ ,  $v_2$  and  $v_3$ . The discussion is somehow similar to the 1-inner case. The contraction of  $e$  yields three connected vertices. Using the similar routine explained above, one finds that each of the bubbles  $\mathbf{b}_{aa_ia_j}$ ,  $i < j$  gets split into two parts after the contraction. Each of these pairs of parts are not related at all by any edge or bow. Thus each bubble produces two bubbles. The resulting bubbles may be open or closed depending on the nature of  $\mathbf{b}_{aa_ia_j}$  in each sector. We have  $B'_{\text{int}} + B'_{\text{ext}} = B_{\text{int}} + B_{\text{ext}} + 3$ .  $\square$

It is clear that the contraction of a 2-inner self-loop also obeys (62) as well for  $p = 2$ . In fact (61) is a stronger result because it mainly separates the variation of the numbers of closed and

open bubbles. For a general 1-inner self-loop contraction, it turns out that the number  $B_{\text{int}} + B_{\text{ext}}$  may vary of 1 or not. Similarly, for a non necessarily trivial 0-inner self-loop contraction, the same number of bubbles may vary from 0,1,2 up to 3.

LEMMA 7 (Faces of a bridge). • *The faces passing through a bridge of any representative of a rank 3 w-colored graph are necessarily open.* • *These faces belong to the same connected component of the boundary graph of the representative.*

PROOF. A bridge  $e$  with color  $a$  has three strands of color pairs  $(aa_1)$ ,  $(aa_2)$  and  $(aa_3)$  defining three faces passing through  $e$ .

The first point is quite clear: following a strand inside the graph, for instance, on the left side of the bridge, before closing that face one has to go on the other side of the bridge without passing by  $e$  (we recall that by the presence of strands with different color pair, we cannot pass twice by a bridge). This is impossible by definition of a bridge. Hence each strand should terminate inside of each side of the bridge hence are open.

Assume that one the faces, say  $(aa_1)$ , does not belong to the same connected component of the boundary of the others. Consider in its own connected component all boundary vertices. Each of these vertices are of valence 3 equipped with edges with different color pairs  $(a_i a_j)$ . Since  $(aa_1)$  should connect one of these boundary vertices, there exists at least one vertex with half edges of color  $a$  or  $a_1$ . Consider all possible vertices where the strand  $(aa_1)$  could connect. There are  $N_a$  and  $N_{a_1}$  such vertices whereas there are  $N_{a_2}$  and  $N_{a_3}$  vertices which cannot be joined by  $(aa_1)$ . Each of these vertices have  $N_{a_\alpha}$ ,  $\alpha = \emptyset, 1, 2, 3$ , half edges of each color pair  $(a_\alpha a_j)$ ,  $j \neq \alpha$ . Since all these half-edges should be fully paired in order to have a connected component therefore each of the following sums should give an even positive integer

$$N_a + N_{a_1} + 1, \quad N_a + N_{a_j}, \quad N_{a_1} + N_{a_j}, \quad j = 2, 3. \quad (63)$$

This system is clearly inconsistent. □

LEMMA 8 (Cut/contraction of special edges). *Let  $\mathcal{G}$  be a representative of  $\mathfrak{G}$  a rank 3 w-colored graph and  $e$  an edge in  $\mathcal{G}$ . Then,*

- *if  $e$  is a bridge, in the above notations, we have*

$$k(\mathcal{G} \vee e) = k(\mathcal{G}/e) + 1, \quad V(\mathcal{G} \vee e) = V(\mathcal{G}/e) + 1, \quad E(\mathcal{G} \vee e) = E(\mathcal{G}/e), \quad f(\mathcal{G} \vee e) = f(\mathcal{G}/e) + 2, \quad (64)$$

$$F_{\text{int}}(\mathcal{G} \vee e) = F_{\text{int}}(\mathcal{G}/e), \quad B_{\text{int}}(\mathcal{G} \vee e) = B_{\text{int}}(\mathcal{G}/e), \quad (65)$$

$$C_{\partial}(\mathcal{G} \vee e) = C_{\partial}(\mathcal{G}/e) + 1, \quad E_{\partial}(\mathcal{G} \vee e) = E_{\partial}(\mathcal{G}/e) + 3, \quad F_{\partial}(\mathcal{G} \vee e) = F_{\partial}(\mathcal{G}/e) + 3, \quad (66)$$

$$B_{\text{ext}}(\mathcal{G} \vee e) = B_{\text{ext}}(\mathcal{G}/e) + 3; \quad (67)$$

- *if  $e$  is a trivial  $p$ -inner self-loop,  $p = 0, 1, 2$ , we have*

$$k(\mathcal{G} \vee e) = k(\mathcal{G}/e) - 2, \quad V(\mathcal{G} \vee e) = V(\mathcal{G}/e) - 2, \quad E(\mathcal{G} \vee e) = E(\mathcal{G}/e), \quad f(\mathcal{G} \vee e) = f(\mathcal{G}/e) + 2, \quad (68)$$

$$F_{\text{int}}(\mathcal{G} \vee e) + C_{\partial}(\mathcal{G} \vee e) = F_{\text{int}}(\mathcal{G}/e) + C_{\partial}(\mathcal{G}/e) - 2, \quad E_{\partial}(\mathcal{G} \vee e) = E_{\partial}(\mathcal{G}/e) + 3, \quad (69)$$

$$B_{\text{int}}(\mathcal{G} \vee e) + B_{\text{ext}}(\mathcal{G} \vee e) = B_{\text{int}}(\mathcal{G}/e) + B_{\text{ext}}(\mathcal{G}/e) - (3 - 2p). \quad (70)$$

Moreover, given  $\partial\mathcal{G}$  the boundary of  $\mathcal{G}$  and a bridge or trivial  $p$ -inner self-loop  $e$ ,  $p = 0, 1, 2, 3$ ,

$$2C_{\partial}(\mathcal{G} \vee e) - \chi(\partial(\mathcal{G} \vee e)) = 2C_{\partial}(\mathcal{G}) - \chi(\mathcal{G}) = 2C_{\partial}(\mathcal{G}/e) - \chi(\partial(\mathcal{G}/e)), \quad (71)$$

where  $\chi(\partial\mathcal{G})$  denote the Euler characteristics of the boundary of  $\mathcal{G}$ .

PROOF. We start by the bridge case. The equations in (64) are easily found. Let us focus on (65). By Lemma 7, we know that necessarily the faces passing through  $e$  are open. All closed faces on each side of the bridge are conserved after cutting  $e$ . The same are still conserved after edge contraction. Hence  $F_{\text{int}}(\mathcal{G} \vee e) = F_{\text{int}}(\mathcal{G}/e)$  and  $B_{\text{int}}(\mathcal{G} \vee e) = B_{\text{int}}(\mathcal{G}/e)$ . We now prove (66). Still by the second point of Lemma 7, the three external faces belong to the same boundary component. After cutting  $e$ , this unique component yields two boundary components. It is direct

to get  $C_\partial(\mathcal{G} \vee e) = C_\partial(\mathcal{G}/e) + 1$ ,  $E_\partial(\mathcal{G} \vee e) = E_\partial(\mathcal{G}/e) + 3$  (the cut of  $e$  divides each external face into two different external strands) and  $F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}/e) + 3$  because  $C_\partial(\mathcal{G}/e) = C_\partial(\mathcal{G})$ ,  $E_\partial(\mathcal{G}/e) = E_\partial(\mathcal{G})$  and  $F_\partial(\mathcal{G}/e) = F_\partial(\mathcal{G})$  which are immediate from Lemma 1. Concerning the number of external bubbles, there are three bubbles in  $\mathcal{G}$  passing through the bridge. Each of these is associated with two color pairs  $(aa_i; aa_j)$ ,  $i < j$ . These bubbles are clearly in  $\mathcal{G}/e$  and cutting the bridge each of these bubble splits in two. This yields (67).

We focus now on a trivial  $p$ -inner self-loop  $e$ . The relations (68) can be determined without difficulty. We concentrate on the rest of the equations. Consider the faces  $f_i$  in  $e$  made with outer strands. For  $p = 0, 1, 2$ , we have  $f_i$ ,  $1 \leq i \leq 3 - p$ . These faces can be open or closed. We do a case by case study according to the number of open or closed faces among the  $f_i$ 's.

- Assume that  $3 - p$  of  $f_i$ 's are closed. Cutting  $e$  entails

$$\begin{aligned} F_{\text{int}}(\mathcal{G} \vee e) &= F_{\text{int}}(\mathcal{G}) - 3, & C_\partial(\mathcal{G} \vee e) &= C_\partial(\mathcal{G}) + 1, & E_\partial(\mathcal{G} \vee e) &= E_\partial(\mathcal{G}) + 3, \\ B_{\text{int}}(\mathcal{G} \vee e) + B_{\text{ext}}(\mathcal{G} \vee e) &= B_{\text{int}}(\mathcal{G}) + B_{\text{ext}}(\mathcal{G}), \end{aligned} \quad (72)$$

$$F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}) + 3. \quad (73)$$

Note that in this situation only the variation of the total number of bubbles can be known.

- Assume that  $3 - p - 1$  of the  $f_i$ 's are closed and one is open. Cutting  $e$  gives

$$\begin{aligned} F_{\text{int}}(\mathcal{G} \vee e) &= F_{\text{int}}(\mathcal{G}) - 2, & C_\partial(\mathcal{G} \vee e) &= C_\partial(\mathcal{G}), & E_\partial(\mathcal{G} \vee e) &= E_\partial(\mathcal{G}) + 3, \\ B_{\text{int}}(\mathcal{G} \vee e) &= B_{\text{int}}(\mathcal{G}) - 1, & B_{\text{ext}}(\mathcal{G} \vee e) &= B_{\text{ext}}(\mathcal{G}) + 1, \end{aligned} \quad (74)$$

$$F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}) + 1. \quad (75)$$

- Assume that  $3 - p - 2$  of the  $f_i$ 's are closed and two are open. Cutting  $e$  gives

$$\begin{aligned} F_{\text{int}}(\mathcal{G} \vee e) &= F_{\text{int}}(\mathcal{G}) - 1, & C_\partial(\mathcal{G} \vee e) &= C_\partial(\mathcal{G}) - 1, & E_\partial(\mathcal{G} \vee e) &= E_\partial(\mathcal{G}) + 3, \\ B_{\text{int}}(\mathcal{G} \vee e) &= B_{\text{int}}(\mathcal{G}), & B_{\text{ext}}(\mathcal{G} \vee e) &= B_{\text{ext}}(\mathcal{G}), \end{aligned} \quad (76)$$

$$F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}) - 1. \quad (77)$$

Note that this case does not apply for  $p = 2$ .

- For  $p = 0$  an additional situation applies: assume that all three  $f_i$ 's are open. Cutting  $e$  gives

$$\begin{aligned} F_{\text{int}}(\mathcal{G} \vee e) &= F_{\text{int}}(\mathcal{G}), & C_\partial(\mathcal{G} \vee e) &= C_\partial(\mathcal{G}) - 2, & E_\partial(\mathcal{G} \vee e) &= E_\partial(\mathcal{G}) + 3, \\ B_{\text{int}}(\mathcal{G} \vee e) &= B_{\text{int}}(\mathcal{G}), & B_{\text{ext}}(\mathcal{G} \vee e) &= B_{\text{ext}}(\mathcal{G}), \end{aligned} \quad (78)$$

$$F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}) - 3. \quad (79)$$

Lemma 6 relates the same numbers for  $\mathcal{G}/e$  and  $\mathcal{G}$  from which one is able to prove (69) and (70).

Last, we prove the relation (71) on the Euler characteristics of the different boundaries. We first note that, from (64), (66), (68) and (69), for any special (bridge or trivial  $p$ -inner,  $p = 0, 1, 2$ ) edge,

$$f(\mathcal{G} \vee e) = f(\mathcal{G}) + 2 = f(\mathcal{G}/e) + 2, \quad E_\partial(\mathcal{G} \vee e) = E_\partial(\mathcal{G}) + 3 = E_\partial(\mathcal{G}/e) + 3. \quad (80)$$

For the bridge case, (71) follows from the relations  $F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}) + 3 = F_\partial(\mathcal{G}/e) + 3$  and  $C_\partial(\mathcal{G} \vee e) = C_\partial(\mathcal{G}) + 1 = C_\partial(\mathcal{G}/e) + 1$  in (66). The result holds also for trivial 0, 1, 2-inner self-loops, after the case by case study giving (72)–(79). Last, for the 3-inner self-loop, in addition to (80) which still holds, the following relations are valid

$$C_\partial(\mathcal{G} \vee e) = C_\partial(\mathcal{G}) + 1 = C_\partial(\mathcal{G}/e) + 1, \quad F_\partial(\mathcal{G} \vee e) = F_\partial(\mathcal{G}) + 3 = F_\partial(\mathcal{G}/e) + 3, \quad (81)$$

and allow us to conclude. □

**4.3. Polynomial invariant for 3D w-colored tensor graphs.** We shall define first an invariant for rank 3 w-colored graphs, check its consistency and then state our main result.

**DEFINITION 36** (Topological invariant for rank 3 w-colored graph). *Let  $\mathfrak{G}(\mathcal{V}, \mathcal{E}, \mathfrak{f}^0)$  be a rank 3 w-colored stranded graph with flags. The generalized topological invariant associated with  $\mathfrak{G}$  is given by the following function associated with any of its representatives  $\mathcal{G}$  (using the above notations)*

$$\begin{aligned} \mathfrak{T}_{\mathfrak{G}}(x, y, z, s, w, q, t) &= \mathfrak{T}_{\mathcal{G};a}(x, y, z, s, w, q, t) = \\ &\sum_{A \in \mathcal{G}} (x-1)^{r(\mathcal{G})-r(A)} (y-1)^{n(A)} z^{5k(A)-[3(V-E(A))+2(F_{\text{int}}(A)-B_{\text{int}}(A)-B_{\text{ext}}(A))]} \\ &\times s^{C_{\partial}(A)} w^{F_{\partial}(A)} q^{E_{\partial}(A)} t^{f(A)}. \end{aligned} \quad (82)$$

Our initial task is to show that the above definition does not actually depend on the representative  $\mathcal{G}$  of  $\mathfrak{G}$ . Given any other representative  $\mathcal{G}'$  of  $\mathfrak{G}$ , there is a one to one map between spanning c-subgraphs of  $\mathcal{G}$  and those of  $\mathcal{G}'$ . This can be achieved by mapping any  $A \in \mathcal{G}$  onto  $A' \in \mathcal{G}'$  such that  $A' = (A \setminus D_{\mathcal{G}}) \cup D_{\mathcal{G}'}$  which can be easily inverted. Now, one checks that  $r(\mathcal{G})$ ,  $r(A)$ ,  $n(A)$  and  $5k(A) - (3V + 2F_{\text{int}}(A))$  do not depend of the representative. The remaining exponents  $B_{\text{int}}(A)$ ,  $B_{\text{ext}}(A)$ ,  $C_{\partial}(A)$ ,  $f(A)$ ,  $E_{\partial}(A)$  and  $F_{\partial}(A)$  depend only on  $A \setminus D_{\mathcal{G}} = A' \setminus D_{\mathcal{G}'}$  hence do not depend on the representative.

One notices that the above consistency checking establishes that only the class of the spanning c-subgraphs of some representative matters in the definition of  $\mathfrak{T}_{\mathfrak{G}}$ . Thus, we could have introduced a state sum on  $\mathfrak{g} \in \mathfrak{G}$  in order to define this invariant directly from the equivalent classes point of view using Proposition 11. We will keep however the more explicit formulation in terms of the spanning c-subgraphs of representatives.

Another crucial point is to show that the exponent of  $z$  in (82) is always a non negative integer.

**PROPOSITION 12.** *Let  $\mathcal{G}$  be any representative of  $\mathfrak{G}$  a rank 3 w-colored graph. Then*

$$\zeta(\mathcal{G}) = 3(E(\mathcal{G}) - V(\mathcal{G})) + 2[B_{\text{int}}(\mathcal{G}) + B_{\text{ext}}(\mathcal{G}) - F_{\text{int}}(\mathcal{G})] \geq 0. \quad (83)$$

**PROOF.** Let us consider the set of closed and open bubbles  $\mathcal{B}_{\text{int}}$  and  $\mathcal{B}_{\text{ext}}$  of  $\mathcal{G}$ , respectively (in this proof, we drop the dependence in the graph  $\mathcal{G}$  in all quantities). Let  $B_{\text{int}}$  and  $B_{\text{ext}}$  be their respective cardinal. Each open or closed bubble  $\mathbf{b}$  is an open or a closed ribbon graph with  $V_{\mathbf{b}}$  number of vertices,  $E_{\mathbf{b}}$  number of edges,  $F_{\text{int};\mathbf{b}}$  number of closed faces and  $C_{\partial}(\mathbf{b})$  number of closed faces obtained uniquely from open strands after pinching (coinciding with the number of connected component of the boundary of the open bubble  $\mathbf{b}$ ).

Any  $\mathbf{b}_i \in \mathcal{B}_{\text{int}}$  being a closed ribbon graph, its Euler characteristics writes

$$2 - \kappa_{\mathbf{b}_i} = V_{\mathbf{b}_i} - E_{\mathbf{b}_i} + F_{\text{int};\mathbf{b}_i}, \quad (84)$$

where  $\kappa_{\mathbf{b}_i}$  refers to the genus of  $\mathbf{b}_i$  or twice its genus if  $\mathbf{b}_i$  is oriented. Summing over all internal bubbles, we get

$$2B_{\text{int}} - \sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} \kappa_{\mathbf{b}_i} = \sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} [V_{\mathbf{b}_i} - E_{\mathbf{b}_i} + F_{\text{int};\mathbf{b}_i}]. \quad (85)$$

Using the colors, one observes that each edge of  $\mathcal{G}$  splits into three ribbon edges belonging either to an open or a closed bubble, and each internal face of  $\mathcal{G}$  belongs to two bubbles which might be open or closed. Thus we have

$$\sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} E_{\mathbf{b}_i} + \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} E_{\mathbf{b}_x} = 3E, \quad \sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} F_{\text{int};\mathbf{b}_i} + \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} F_{\text{int};\mathbf{b}_x} = 2F_{\text{int}}. \quad (86)$$

In addition, each vertex of the graph can be decomposed, at least, in three vertices (3 vertices is the minimum given by the simplest vertex of the form  $\mathcal{G}_1$  in Figure 29) which could belong to an open or closed bubble, we have

$$\sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} V_{\mathbf{b}_i} + \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} V_{\mathbf{b}_x} \geq 3V. \quad (87)$$

Combining (86) and (87), we re-write (85) as

$$3V - 3E + 2F_{\text{int}} - 2B_{\text{int}} - \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} [V_{\mathbf{b}_x} - E_{\mathbf{b}_x} + F_{\text{int}; \mathbf{b}_x}] \leq - \sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} \kappa_{\mathbf{b}_i}. \quad (88)$$

We complete the last sum involving  $\mathcal{B}_{\text{ext}}$  by adding  $C_{\partial}(\mathbf{b}_x)$  in order to get

$$\sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} [V_{\mathbf{b}_x} - E_{\mathbf{b}_x} + F_{\text{int}; \mathbf{b}_x} + C_{\partial}(\mathbf{b}_x)] = \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} (2 - \kappa_{\tilde{\mathbf{b}}_x}), \quad (89)$$

which, substituted in (88), leads us to

$$3V - 3E + 2F_{\text{int}} - 2B_{\text{int}} - 2B_{\text{ext}} \leq - \sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} \kappa_{\mathbf{b}_i} - \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} (C_{\partial}(\mathbf{b}_x) + \kappa_{\tilde{\mathbf{b}}_x}) \quad (90)$$

from which the lemma results.  $\square$

In fact,  $\sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} C_{\partial}(\mathbf{b}_x) = F_{\partial}(\mathcal{G})$  is the number of faces of the boundary graph  $\mathcal{G}$  (see Remark 1). The above bound can be refined since  $F_{\partial}(\mathcal{G}) \geq B_{\text{ext}}(\mathcal{G})$  which merely follows from the fact that each  $\mathbf{b}_x \in \mathcal{B}_{\text{ext}}$  has at least a boundary contributing to  $F_{\partial}(\mathcal{G})$  such that

$$B_{\text{ext}}(\mathcal{G}) \leq \sum_{\mathbf{b}_x} C_{\partial}(\mathbf{b}_x) = F_{\partial}(\mathcal{G}). \quad (91)$$

Thus we also have

$$3V - 3E + 2F_{\text{int}} - 2B_{\text{int}} - B_{\text{ext}} \leq - \sum_{\mathbf{b}_i \in \mathcal{B}_{\text{int}}} \kappa_{\mathbf{b}_i} - \sum_{\mathbf{b}_x \in \mathcal{B}_{\text{ext}}} \kappa_{\tilde{\mathbf{b}}_x}. \quad (92)$$

This may lead as well to yet another well defined invariant. However, we will not use this relation in the following due to some rich relations that  $-\zeta(A) \geq 0$  entails. In particular, we have other positive combinations.

**PROPOSITION 13.** *Let  $\mathcal{G}$  be any representative of  $\mathfrak{G}$  a rank 3 w-colored graph. Then*

$$\begin{aligned} \zeta'(\mathcal{G}) &= 3[E(\mathcal{G}) - V(\mathcal{G})] + 2[B_{\text{int}}(\mathcal{G}) + B_{\text{ext}}(\mathcal{G}) - F_{\text{int}}(\mathcal{G}) - C_{\partial}(\mathcal{G})] \geq 0, \\ \zeta''(\mathcal{G}) &= 3[E(\mathcal{G}) - V(\mathcal{G}) - C_{\partial}(\mathcal{G})] + 2[B_{\text{int}}(\mathcal{G}) + B_{\text{ext}}(\mathcal{G}) - F_{\text{int}}(\mathcal{G})] \geq 0. \end{aligned} \quad (93)$$

**PROOF.** By the color prescription, each connected component of the boundary of  $\mathcal{G}$  has at least three faces. Therefore

$$3C_{\partial} \leq F_{\partial}. \quad (94)$$

Then we also have  $-F_{\partial} + 2C_{\partial} \leq 0$ . The proposition follows from (90) and the fact that  $F_{\partial} = \sum_{\mathbf{b}_x} C_{\partial}(\mathbf{b}_x)$  in the proof of Proposition 12.  $\square$

Proposition 12 ensures  $\zeta(A) \geq 0$ , for any  $A \in \mathcal{G}$ , hence the following statement.

**PROPOSITION 14** (Polynomial invariant).  *$\mathfrak{T}_{\mathfrak{G}} = \mathfrak{T}_{\mathcal{G}}$  is a polynomial.*

The quantity  $V - E(A) + F_{\text{int}}(A) - B_{\text{int}}(A)$  is nothing but the Euler characteristics for a closed colored graph  $A$  understood as a cellular complex [10]. This quantity always proves to be bounded by  $-\sum_{\mathbf{b}_i} \kappa_{\mathbf{b}_i}$ . If the graph is not closed, the same cellular complex has a boundary bearing itself a cellular decomposition. The quantity  $-\zeta(A)$  represents, in the present specific instance, a weighted notion for Euler characteristics of the cellular complex corresponding to  $A$  which also take into account the boundary due to  $B_{\text{ext}}$ . The quantity  $5k(A) - \zeta(A)$  will be however the one of interest in the following.

Let us call  $\mathcal{G}^0$  a representative graph made only with a finite family of vertices with flags and without any edges possibly with discs. Then  $E(\mathcal{G}^0) = B_{\text{int}}(\mathcal{G}^0) = 0$ ,  $F_{\text{int}}(\mathcal{G}^0) = D$ ,  $D$  being the number of trivial discs in  $\mathcal{G}^0$ ,  $k(\mathcal{G}^0) = V(\mathcal{G}^0)$ ,  $k(\mathcal{G}^0) - D = C_{\partial}(\mathcal{G}^0) \geq 0$ , we can check the consistency of (82) as the following makes still a sense:

$$\mathfrak{T}_{\mathcal{G}^0}(x, y, z, s, w, q, t) = z^{2[k(\mathcal{G}^0) - D + B_{\text{ext}}(\mathcal{G}^0)]} s^{k(\mathcal{G}^0) - D} w^{F_{\partial}(\mathcal{G}^0)} q^{E_{\partial}(\mathcal{G}^0)} t^{|f^0(\mathcal{G}^0)|}. \quad (95)$$

It may exist several possible reductions of the above polynomial. We will however focus on the most prominent ones given by

$$\begin{aligned}\mathfrak{T}_{\mathcal{G}}(x, y, z, z^{-2}, w, q, t) &= \mathfrak{T}'_{\mathcal{G}}(x, y, z, w, q, t), & \mathfrak{T}_{\mathcal{G}}(x, y, z, z^{-2}s^2, s^{-1}, s, s^{-1}) &= \mathfrak{T}''_{\mathcal{G}}(x, y, z, s), \\ \mathfrak{T}_{\mathcal{G}}(x, y, z, z^2z^{-2}, z^{-1}, z, z^{-1}) &= \mathfrak{T}'''_{\mathcal{G}}(x, y, z).\end{aligned}\tag{96}$$

Proposition 13 ensures that  $\mathfrak{T}'_{\mathcal{G}}$  is a polynomial. Meanwhile,  $\mathfrak{T}''_{\mathcal{G}}$  is also a polynomial with exponent of  $s$  the Euler characteristics of the boundary  $\partial\mathcal{G}$ .  $\mathfrak{T}'''_{\mathcal{G}}(x, y, z)$  combine both invariants in a single exponent. These polynomials will be relevant in the next analysis. Note that we could have introduced another polynomial between  $\mathfrak{T}'$  and  $\mathfrak{T}''$  which expresses as  $\mathfrak{T}_{\mathcal{G}}(x, y, z, s^2, s^{-1}, s, s^{-1}) = \mathfrak{T}^0_{\mathcal{G}}(x, y, z, s)$ . But this latter turns out to satisfy the same properties as  $\mathfrak{T}$  and thus does not provide anything new.

We are now in position to prove our main theorem.

**THEOREM 4** (Contraction/cut rule for  $w$ -colored graphs). *Let  $\mathfrak{G}$  be a rank 3  $w$ -colored graph with flags. Then, for a regular edge  $e$  of any of the representative  $\mathcal{G}$  of  $\mathfrak{G}$ , we have*

$$\mathfrak{T}_{\mathcal{G}} = \mathfrak{T}_{\mathcal{G} \vee e} + \mathfrak{T}_{\mathcal{G}/e}, \tag{97}$$

for a bridge  $e$ , we have  $\mathfrak{T}_{\mathcal{G} \vee e} = z^8 s(wq)^3 t^2 \mathfrak{T}_{\mathcal{G}/e}$  and

$$\mathfrak{T}_{\mathcal{G}} = [(x-1)z^8 s(wq)^3 t^2 + 1] \mathfrak{T}_{\mathcal{G}/e}; \tag{98}$$

for a trivial  $p$ -inner self-loop  $e$ ,  $p = 0, 1, 2$ , we have

$$\mathfrak{T}_{\mathcal{G}} = \mathfrak{T}_{\mathcal{G} \vee e} + (y-1)z^{4p-7} \mathfrak{T}_{\mathcal{G}/e}. \tag{99}$$

**PROOF.** Let  $\mathcal{G}$  be any representative of the class  $\mathfrak{G}$ . Same initial remarks as stated in the proof of Theorem 3 hold here for  $\mathcal{G}$ . Our main concern is the change in the number of internal and external bubbles and the independence of the final result on the representative  $\mathcal{G}$ .

We concentrate first on (97). Consider an ordinary edge  $e$  of  $\mathcal{G}$ , the set of spanning  $c$ -subgraphs which do not contain  $e$  being the same as the set of spanning  $c$ -subgraphs of  $\mathcal{G} \vee e$ , the number of open and closed bubbles on each subgraph is the same, it is direct to get  $\sum_{A \in \mathcal{G}; e \notin A} (\cdot) = \mathfrak{T}_{\mathcal{G} \vee e}$ . Observe that the class of  $\mathcal{G} \vee e$  defines  $\mathfrak{G} \vee e$  from Lemma 5 and we know that all exponents in  $\mathfrak{T}_{\tilde{\mathcal{G}}}$  do not depend on the representative  $\tilde{\mathcal{G}}$  of the class of  $\mathfrak{G}$ . Hence  $\mathfrak{T}_{\mathcal{G} \vee e} = \mathfrak{T}_{\mathfrak{G} \vee e}$  does not depend on the representative.

Let us focus on the second term of (97). Consider  $e$ , its end vertices  $v_1$  and  $v_2$  and its 3 strands with color pairs  $(aa_1)$ ,  $(aa_2)$  and  $(aa_3)$  and consider the set bubbles in  $\mathfrak{G}$ . Some bubbles do not use any strand of  $e$  and three bubbles can be formed using these strands (these are of colors  $(aa_i a_j)$ ,  $i < j$ ). Contracting now  $e$ , the vertex obtained is connected by Lemma 4. The former three strands  $e$  are clearly preserved after the contraction. The bubbles not passing through  $e$  are not affected at all by the procedure. The three bubbles passing through  $e$  are also preserved since the contraction do not delete faces or strands. The faces passing through  $e$  getting only shortened, the net result from the point of view of the bubbles passing through  $e$  is simply an ordinary ribbon edge contraction in the sense of ribbon graphs. From this results  $\sum_{A \in \mathcal{G}; e \in A} (\cdot) = \mathfrak{T}_{\mathcal{G}/e}$ . By Lemma 5, we define  $\mathfrak{G}/e$  and conclude that the polynomial  $\mathfrak{T}_{\mathfrak{G}/e}$  does not depend on the representative  $\mathcal{G}/e$  of  $\mathfrak{G}/e$ .

Let us focus now on the bridge case and (98). Cutting a bridge yields, as in the ordinary case, from the sum  $\sum_{A \in \mathcal{G}; e \notin A}$  the product  $(x-1)\mathfrak{T}_{\mathcal{G} \vee e}$ . The second sum remains as it is using the mapping between  $\{A \in \mathcal{G}; e \in A\}$  and  $\{A \in \mathcal{G}/e\}$  and provided the fact that there is no change in the vertex contracted which is ensured by Lemma 4. The last stage relates  $\mathfrak{T}_{\mathcal{G} \vee e}$  and  $\mathfrak{T}_{\mathcal{G}/e}$ . This can be achieved by using the bijection between  $A \in \mathcal{G} \vee e$  and  $A' \in \mathcal{G}/e$  where each  $A$  and  $A'$  are both uniquely related to some  $A_0 \in \mathcal{G}$  as  $A = A_0 \vee e$  and  $A' = A_0/e$ . Using Lemma 8, the relation (98) follows. Then, Lemma 5 allows us to define  $(\mathfrak{G} \vee e)$  and  $\mathfrak{G}/e$  and to conclude that the resulting polynomials  $(\mathfrak{T}_{\mathcal{G} \vee e})$  and  $\mathfrak{T}_{\mathcal{G}/e}$  are independent on the representative  $(\mathcal{G} \vee e)$  and  $\mathcal{G}/e$ .

Next, we discuss the trivial  $p$ -inner self-loop case and prove (99). The property that the sum  $\sum_{A \in \mathcal{G}; e \notin A} (\cdot) = \mathfrak{T}_{\mathcal{G} \vee e}$  should be direct. The second sum is now studied.



The question is whether or not  $e$  being a trivial  $p$ -inner self-loop in  $\mathcal{G}$  remains as such in  $A$ . The answer for that question is yes because  $A$  contains the end vertex of  $e$ . We may cut some edges in each sectors  $v_i$  (see Figure 30) for defining  $A$  but the resulting sectors are still totally separated in  $A$ .

Contracting a trivial  $p$ -inner self-loop generates  $p$  discs and  $3 - p$  non trivial vertices. Now, from Lemma 6, we know how evolve all numbers of components in the graph: the nullity is again  $n(A) = n(A') + 1$ , and the exponent of  $z$  becomes

$$\begin{aligned} & 5k(A) - (3(V(\mathcal{G}) - E(A)) + 2(F_{\text{int}}(A) - B_{\text{int}}(A) - B_{\text{ext}}(A))) = \\ & 5(k(A') - 2) - \left[ 3[V(\mathcal{G}/e) - 2] - (E(A') + 1) + 2(F_{\text{int}}(A') - B_{\text{int}}(A') - B_{\text{ext}}(A') + \alpha_p) \right] \\ & = 5k(A') - (3(V(\mathcal{G}/e) - E(A')) + 2(F_{\text{int}}(A') - B_{\text{int}}(A') - B_{\text{ext}}(A')) - 7 + 4p, \end{aligned} \quad (100)$$

where  $\alpha_p = 3 - 2p$ ,  $A$  and  $A'$  refer to the subgraphs related by bijection between spanning c-subgraphs of  $\{A \in \mathcal{G}; e \in A\}$  and  $\{A' \in \mathcal{G}/e\}$ . At the end, one gets  $(y - 1)z^{4p-7}\mathfrak{T}_{\mathcal{G}/e}$ . In the similar way as done before, we are able to conclude that this result is independent of the representative.  $\square$

Some comments are in order at this point.

- The existence of a function defined on  $\mathfrak{G}$  satisfying the relations of Theorem 4 is guaranteed by the explicit formula of  $\mathfrak{T}_{\mathfrak{G}}$  given in Definition 36. The unicity of  $\mathfrak{T}_{\mathfrak{G}}$  follows from its existence and can be only guaranteed by suitable “boundary condition” data which are given by the bridge contraction (98) and the data of all one-vertex (rosette) w-colored graph polynomials. These polynomials write again using the state sum (sum over subgraphs) of such stranded rosette graphs (82). These conditions extend boundary conditions making unique the BR polynomial [6].

- Note that in the equality (100) the number of bubbles  $B_{\text{int}} + B_{\text{ext}}$  plays a crucial role in the determination of the exponent of  $z$ . In addition, the contraction of a bridge for this category of graph with flags is close to the Tutte/BR relation for bridge contraction since we do not need any constrain in the polynomial (for instance putting  $s = z^{-1}$  in the case of the BR polynomial for graphs with flags) in order to obtain a one term relation for such a terminal form. For the last contraction/cut for a trivial  $p$ -inner self-loop (99), we still cannot map  $\mathfrak{T}_{\mathcal{G} \vee e}$  on  $\mathfrak{T}_{\mathcal{G}/e}$ . These features show that the polynomial  $\mathfrak{T}$  is a genuine extension of the BR polynomial.

**COROLLARY 2** (Cut/contraction relations for  $\mathfrak{T}'$ ). *Let  $\mathfrak{G}$  be a rank 3 w-colored stranded graph with flags and any of its representative  $\mathcal{G}$ . Then, for a bridge  $e$ , we have  $\mathfrak{T}'_{\mathcal{G} \vee e} = z^6(wq)^3 t^2 \mathfrak{T}'_{\mathcal{G}/e}$  and*

$$\mathfrak{T}'_{\mathcal{G}}(x, y, z, w, q, t) = [(x - 1)z^6(qw)^3 t^2 + 1] \mathfrak{T}'_{\mathcal{G}/e}(x, y, z, w, q, t); \quad (101)$$

for a trivial  $p$ -inner self-loop  $e$  in  $\mathcal{G}$ ,  $0 \leq p \leq 2$ , we have

$$\mathfrak{T}'_{\mathcal{G}}(x, y, z, 1, q, t) = z^{4p-6} [q^3 t^2 + (y - 1)z^{-1}] \mathfrak{T}'_{\mathcal{G}/e}(x, y, z, 1, q, t); \quad (102)$$

**PROOF.** Theorem 4 implies naturally (101) after the setting  $s = z^{-2}$  in  $\mathfrak{T}_{\mathcal{G}}$ . We now work out the contraction/cut of the trivial self-loops.

Same arguments as in the proof of Theorem 4 should be invoked here. The difference now is that by changing  $s \rightarrow z^{-2}$ , using (1) the one-to-one mapping between  $\{A \in \mathcal{G} \vee e\}$  and  $\{A' \in \mathcal{G}/e\}$  such that to each  $A \in \mathcal{G} \vee e$  one associates  $A' \in \mathcal{G}/e$  defined by  $A' = \tilde{A}/e$ ,  $\tilde{A} \in \mathcal{G}$ , and  $\tilde{A} \vee e = A$  and (2) the relations (68)-(70) of Lemma 8, we can map, for a trivial 0, 1, 2-inner self-loop  $e$ ,  $\mathfrak{T}_{\mathcal{G} \vee e}$  on  $\mathfrak{T}_{\mathcal{G}/e}$ . We compute the variation of the exponent of  $z$  between  $A$  and  $A'$  as

$$\begin{aligned} & 5k(A) - (3(V(\mathcal{G}) - E(A)) + 2[F_{\text{int}}(A) + C_{\partial}(A) - B_{\text{int}}(A) - B_{\text{ext}}(A)]) = \\ & 5(k(A') - 2) - (3(V(\mathcal{G}/e) - 2 - E(A')) + 2[F_{\text{int}}(A') + C_{\partial}(A') - 2 - B_{\text{int}}(A') - B_{\text{ext}}(A') + \alpha_p]) \\ & = 5k(A') - (3(V(\mathcal{G}/e) - E(A')) + 2[F_{\text{int}}(A') - B_{\text{int}}(A') - B_{\text{ext}}(A')]) - 2\alpha_p. \end{aligned} \quad (103)$$

The rest of the variations can be easily identified using the same lemma. It can be proved that the result does not depend on the representative.  $\square$

As a comment, for  $w \neq 1$ ,  $\mathfrak{T}'_{\mathcal{G} \vee e}$  cannot be fully mapped onto  $\mathfrak{T}'_{\mathcal{G}/e}$  due to the fact that the number  $F_\partial$  of face of the boundary of each subgraph cannot be determined in that general situation. Nevertheless, we have two other noteworthy reductions which solve this issue without putting  $w = 1$ .

**COROLLARY 3** (Cut/contraction rules for  $\mathfrak{T}''$  (and  $\mathfrak{T}'''$ )). *Let  $\mathfrak{G}$  be a rank 3  $w$ -colored stranded graph with flags and any of its representative  $\mathcal{G}$ .*

*Then, for a bridge  $e$ , we have  $\mathfrak{T}''_{\mathcal{G} \vee e} = z^6 \mathfrak{T}''_{\mathcal{G}/e}$  and*

$$\mathfrak{T}''_{\mathcal{G}} = [(x-1)z^6 + 1] \mathfrak{T}''_{\mathcal{G}/e}; \quad (104)$$

*for a trivial  $p$ -inner self-loop  $e$  in  $\mathcal{G}$ ,  $0 \leq p \leq 2$ , we have  $\mathfrak{T}''_{\mathcal{G} \vee e} = z^{4p-6} \mathfrak{T}''_{\mathcal{G}/e}$  and*

$$\mathfrak{T}''_{\mathcal{G}} = z^{4p-6} [1 + (y-1)z^{-1}] \mathfrak{T}''_{\mathcal{G}/e}. \quad (105)$$

*The same contraction and cut rules applies for  $\mathfrak{T}'''$ .*

**PROOF.** The new ingredient to achieve the proof of this statement is (71) of Lemma 8.  $\square$

More remarks can be made at this stage.

- $\mathfrak{T}''$  and  $\mathfrak{T}'''$  satisfy the same rules. Hence  $\mathfrak{T}''$  which is more general becomes more relevant.
- We did not use the notion of product of graphs for mapping the polynomials for  $\mathcal{G} \vee e$  onto those of  $\mathcal{G}/e$ . In fact, even for ribbon graphs our approach applies. The dot product of graphs is however an elegant way to encode such a map. Certainly, this notion deserves to be defined in the framework of stranded graphs in order to achieve more calculations. For instance, how the calculation of more involved graphs with many trivial self-loops can be inferred from the product of contributions of much simpler trivial self-loops.

- The exponents of  $z^{4p-7}$  or of  $z^{4p-6}$  can be negative in some expressions. This simply implies that in the polynomial  $\mathfrak{T}'_{\mathcal{G}/e}$  or  $\mathfrak{T}''_{\mathcal{G}/e}$  all monomials should contain an enough large power of  $z$  to render the overall expression a well defined polynomial.

If the notion of point graph multiplication of two disjoint graphs is not defined at this stage, in contrast, the disjoint union operation on graph extends naturally in the present formulation.

**LEMMA 9** (Disjoint union). *Let  $\mathfrak{G}_1$  and  $\mathfrak{G}_2$  two disjoint rank 3  $w$ -colored graphs, then*

$$\mathfrak{T}_{\mathfrak{G}_1 \sqcup \mathfrak{G}_2} = \mathfrak{T}_{\mathfrak{G}_1} \mathfrak{T}_{\mathfrak{G}_2}. \quad (106)$$

**PROOF.** This is totally standard as in the ordinary procedure using additive properties of exponents in  $\mathfrak{T}_{\mathcal{G}}$ .  $\square$

**COROLLARY 4** (3-inner self-loop contraction). *Given a rank 3  $w$ -colored graph  $\mathfrak{G}$  and  $\mathcal{G}$  one of its representative containing a 3-inner self-loop  $e$  then*

$$\mathfrak{T}_{\mathcal{G}} = z^5 (z^3 s(wq)^3 t^2 + y - 1) \mathfrak{T}_{\mathcal{G}/e}. \quad (107)$$

**PROOF.** We use the fact that  $e$  is a 3-inner then it forms a separate subgraph  $g$ . In order to compute the polynomial of  $\mathcal{G}$ , Lemma 9 can be applied and a direct evaluation of the graph  $g$  yields the desired factor.  $\square$

**DEFINITION 37** (Multivariate form). *The multivariate form associated with (82) is defined by:*

$$\begin{aligned} \tilde{\mathfrak{T}}_{\mathfrak{G}}(x, \{\beta_e\}, \{z_i\}_{i=1,2,3}, s, w, q, t) &= \tilde{\mathfrak{T}}_{\mathcal{G}}(x, \{\beta_e\}, \{z_i\}_{i=1,2,3}, s, w, q, t) \\ &= \sum_{A \in \mathcal{G}} x^{r(A)} \left( \prod_{e \in A} \beta_e \right) z_1^{F_{\text{int}}(A)} z_2^{B_{\text{int}}(A)} z_3^{B_{\text{ext}}(A)} s^{C_\partial(A)} w^{F_\partial(A)} q^{E_\partial(A)} t^{f(A)}, \end{aligned} \quad (108)$$

*for  $\{\beta_e\}_{e \in \mathcal{E}}$  labeling the edges of the graph  $\mathcal{G}$ .*

It is direct to prove the following statement by ordinary techniques.

PROPOSITION 15. For all regular edge  $e$ ,

$$\tilde{\mathfrak{T}}_{\mathfrak{G}} = \tilde{\mathfrak{T}}_{\mathfrak{G} \setminus e} + x\beta_e \tilde{\mathfrak{T}}_{\mathfrak{G}/e}. \quad (109)$$

REMARK 2. We can now compare  $\tilde{\mathfrak{T}}$  with Gurau's polynomial denoted in the following by  $G$  [12]. We shall use however not the full form of  $G_{\mathcal{G}}$  denoted  $P_{\mathcal{G}}$  in [12] but we will use instead two improvements: (1) the normalized form  $P_{\mathcal{G}}(\{\beta_e x_1\}, \dots)$  of  $P_{\mathcal{G}}$ , where  $x_1$  is the variable associated with the number of edges which brings an inessential overall factor of  $x_1^{E(A)}$  consistently absorbed by the  $\beta_e$  and (2) a rank formulation of  $G_{\mathcal{G}}(\{\beta_e\}, \dots) = P_{\mathcal{G}}(\{\beta_e x_1\}, \dots)$ , i.e. rather than using two variables  $x_0$  for the vertices and  $x_4$  for the number of connected components of the rank 3 colored tensor graph, we will simply use  $x^{r(A)}$ ,  $A \in \mathcal{G}$ . This notation is more compact and  $r(A)$  is the quantity depending only on the class of  $A$ .

For a rank 3 colored tensor graph  $\mathcal{G}$ , the polynomials  $\tilde{\mathfrak{T}}$  and  $G$  are related by

$$\tilde{\mathfrak{T}}_{\mathcal{G}}(x, \{\beta_e\}, z_1, z_2, z_3 = 1, s, w, q, t) = G_{\mathcal{G}}(x, \{\beta_e\}, z_1, z_2, s, q, w, t), \quad (110)$$

with, according to the convention in [12], we have

$$C_{\partial} = |\mathfrak{B}_{\partial}^3|, \quad F_{\partial} = |\mathfrak{B}_{\partial}^2|, \quad E_{\partial} = |\mathfrak{B}_{\partial}^1|, \quad f = |\mathfrak{B}_{\partial}^0|, \quad B_{\text{int}} = |\mathfrak{B}^3|, \quad F_{\text{int}} = |\mathfrak{B}^2|. \quad (111)$$

As expected, for this category of graphs the polynomial  $\tilde{\mathfrak{T}}$  is more general because it contains one additional variable (denoted above by  $z_3$ ) which is associated with the number of external bubbles of the graph. This variable can be introduced by hand in  $G$  making it a little more general. This additional variable does not lead to any new features for the multivariate form however, as seen in our developments, it plays an important role in the non-multivariate form.

We finally wrap up this section with additional insights:

- We have successfully defined the classes of rank 3 w-colored stranded graphs for which a natural contraction/cut rule makes a sense and the associated polynomial invariants  $\mathfrak{T}$ ,  $\mathfrak{T}'$  and  $\mathfrak{T}''$ . Our results extend both BR polynomial to arbitrary contractions of colored tensor graphs of rank 3. The achievement of this requires to relax and to modify several initial concepts introduced by Gurau (to mention a few, the meaning of 3-bubbles with now no fixed valence and the very notion of contraction itself, for instance). The form of our polynomial is not primarily given in the multivariate form (as the one by Gurau). Thus, it has required a full-fledge proof in order to establish that this polynomial is a genuine invariant for the present graphs. We have also successfully identified terminal forms of the contraction/cut sequence for this type of graphs. Such a study of terminal forms has been never addressed in former works on stranded graphs [12][18].

- We stress that after the contraction of some  $p$ -inner self-loops, other remaining trivial  $p'$ -inner self-loops may change their number of  $p'$  inner faces. This is the reason why given a certain number of  $m_p$  trivial  $p$ -inner self-loops, we cannot easily infer the number of trivial 0, 1, 2- or 3-inner self-loops which shall really contribute to the final evaluation of the terminal form. In fact, we can show that contracting a trivial  $p$ -inner self-loop at some point of the sequence may only be neutral for or increase by 1 the number of inner faces of other  $p$ -inner self-loops. Hence, roughly speaking, after contraction there may or may not have a “migration” of the elements of the set of  $p$ -inner self-loops to set of the  $(p+1)$ -inner self-loops,  $p+1 \leq 3$ .

In any case, the polynomial  $\mathfrak{T}''$  of some terminal form  $\mathcal{G}$  representative of a w-colored graph made only with bridges and trivial  $p$ -inner self-loops can be computed as

$$x^m \left[ \prod_{p=0}^3 \{z^{4p-6}[1 + (y-1)z^{-1}]\}^{n_p} \right] \times \mathfrak{T}''(\mathcal{G}^0), \quad (112)$$

for some integer  $m, n_{0,1,2,3}$ , and where  $\mathfrak{T}''(\mathcal{G}^0)$  is the polynomial of  $\mathcal{G}^0$  graph having only vertices and hence is given by (95). After changing  $s \rightarrow z^{-2}s^2$ , one gets

$$\mathfrak{T}''(\mathcal{G}^0) = z^{2B_{\text{ext}}(\mathcal{G}^0)} s^{2C_{\partial}(\mathcal{G}^0) - (f(\mathcal{G}^0) - E_{\partial}(\mathcal{G}^0) + F_{\partial}(\mathcal{G}^0))}. \quad (113)$$

Note that the contribution (112) only makes a sense as a polynomial if exponents  $n_p$  are not arbitrary. One must have indeed

$$5n_3 + n_2 - 3n_1 - 7n_0 + 2B_{\text{ext}}(\mathcal{G}^0) \geq 0, \quad (114)$$

which should follow from Proposition 12. However it is interesting to show that this is indeed true in a different (and quick) manner. The issue here is to show that the total number of trivial 1-inner and 0-inner contributions cannot exceed the number of 3-inner, 2-inner contributions and external bubbles after the overall contraction. In fact, whenever we contract a trivial 1-inner or 0-inner self-loop, we split the one-vertex graph in  $m_1 = 2$  or  $m_0 = 3$  one-vertex (non discs) graphs and get either a power of  $z^{-3}$  or  $z^{-7}$ , respectively. On each of these resulting graphs, we can pursue the contraction. But, at the end, either we are led to a connected component of the form of a 3-inner self-loop and we gain a power of  $z^5$  per connected component obtained, or to a final vertex with only flags (see Lemma 1). The simplest non trivial vertex-graph with the minimal number of external bubbles is given  $\mathcal{G}_1$  in Figure 29 and yields  $B_{\text{ext}}(\mathcal{G}_1) = 3$ . Call  $\beta_3$  the number of components yielding a 3-inner self-loop and  $\beta_x$  the number of components yielding an empty vertex graph. Then

$$\beta_3 + \beta_x = 2n_1 + 3n_0, \quad \beta_3 \leq n_3, \quad 3\beta_x \leq B_{\text{ext}}(\mathcal{G}^0), \quad (115)$$

and hence

$$\begin{aligned} -3n_1 - 7n_0 + 5n_3 + 2B_{\text{ext}}(\mathcal{G}^0) &\geq -3n_1 - 7n_0 + 5\beta_3 + 2B_{\text{ext}}(\mathcal{G}^0) \\ &\geq 7n_1 + 8n_0 + 2(B_{\text{ext}}(\mathcal{G}^0) - 5\beta_x/2) \geq 0. \end{aligned} \quad (116)$$

In all situation, we can always beat the negative power introduced by the trivial 0- or 1-inner edge.

• Studying the limit cases, we have the following reduced polynomials: for a w-colored graph  $\mathfrak{G}$  and any representative  $\mathcal{G}$  of  $\mathfrak{G}$ ,

$$\mathfrak{T}_{\mathcal{G}}(x, y, z = 1, s = 1, w = 1, q = 1, t) = \mathcal{T}_{\mathcal{G}}(x, y, t), \quad (117)$$

where, computing  $\mathcal{T}$  for  $\mathcal{G}$ , we consider in its simple graph (collapsed) form.

There is a priori no direct mapping between  $\mathfrak{T}$  and BR polynomials (with and without flags) because these two polynomials have a different “dimensionality” or rank with respect to stranded structures. One might think about a combinatorial manipulation by collapsing the tensor structure to a ribbon graph by removing a strand. This deserves to be fully understood. However, in order to find such a mapping with BR polynomials, a natural way would be to define the classes of “rank 2” w-colored graphs and their associated polynomials. These latter should coincide with  $\mathcal{R}$  and  $\mathcal{R}'$  after identifications and reduction of some variables. There will be nothing more general than  $\mathcal{R}$  and  $\mathcal{R}'$  in such context. The point raised in rank  $D \geq 3$  is the presence of  $p \geq 3$ -bubbles.

Few important properties have to be addressed:

• The **universality** property of  $\mathfrak{T}$ ,  $\mathfrak{T}'$  or  $\mathfrak{T}''$  which is the fact that any other function satisfying (97) and (98) can be expressed in terms of one of these polynomial needs to be understood. This task is not performed in this work which has mainly focused on the definition of the relevant graph objects for which a polynomial invariant can be identified. The universality will be addressed in a forthcoming work and will require a complete understanding of the one-vertex w-colored graphs as they appear in the formalism. We think that the introduction of colors will help in reducing the type of graph structures that we need to classify. Hence, we aim at finding suitable classes of graphs for which one of the polynomials  $\mathfrak{T}$ ,  $\mathfrak{T}'$  or  $\mathfrak{T}''$  will prove to be universal.

• In fact, before addressing the universality issue on stranded structures, one first needs to understand to what extent the universality property can be recovered for the extension  $\mathcal{R}$  of the BR polynomial for ribbon graphs with flags (without pinching). This will amount to generalize the notion of chord diagrams  $\mathcal{D}$ , their equivalence relation under rotations about chords and finally their associated basic canonical diagrams  $\mathcal{D}_{ijk}$  (counting the nullity  $i$ , genus  $j$  of the diagram,  $k$  is associated with the orientation of the surface associated with diagram) [6, 5] to chord diagrams with flags. Such a preliminary task will certainly imply the existence of a new canonical diagram

$\mathcal{D}_{ijkl}$ , where  $i, j, k$  keep their meaning and  $l$  defines the number of connected component associated with the boundary of the open ribbon graph.

- There exists another formulation of the Tutte and BR polynomials in terms of spanning tree expansions. Such a notion deserves to be investigated as well in the context of  $w$ -colored stranded graphs.

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### Appendix: Examples

We carry out explicitly two examples in order illustrate our results in the present appendix. We will not focus on the equivalence class  $\mathfrak{G}$  but on a particular representative always denoted by  $\mathcal{G}$ .

**Example 1: A simple colored graph.** Consider the graph  $\mathcal{G}$  given by Figure 32. Computing the multivariate form of the polynomial, we obtain by the spanning c-subgraph summation in (108) with  $\beta_i$  is associated with the edge of color  $i$ ,

$$\begin{aligned} \mathfrak{T}_{\mathcal{G}}(x, \{\beta_e\}, z, s, q, t) &= \beta_0 \beta_1 \beta_2 \beta_3 x z_1^6 z_2^4 \\ &+ (\beta_0 \beta_1 \beta_2 + \beta_0 \beta_2 \beta_3 + \beta_1 \beta_2 \beta_3 + \beta_0 \beta_1 \beta_3) x z_1^3 z_2 z_3^3 s w^3 q^3 t^2 \\ &+ (\beta_0 \beta_1 + \beta_0 \beta_2 + \beta_0 \beta_3 + \beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3) x z_1 z_3^4 s w^4 q^6 t^4 \\ &+ (\beta_0 + \beta_1 + \beta_2 + \beta_3) x z_3^5 s w^5 q^9 t^6 + z_3^8 s^2 w^8 q^{12} t^8. \end{aligned} \quad (\text{A.1})$$

Then this polynomial coincides exactly with the normalized Gurau's polynomial  $G_{\mathcal{G}}$  after setting the additional variable  $z_3$  associated with external bubbles to 1. Note that reversely, introducing a new variable associated with the same component in  $C_{\mathcal{G}}$  both polynomials match, for this example. Note also that, in this specific example the exponent of  $z_3$  and of  $w$  always coincide. However it is not always true that each open bubble would have a unique boundary component.

Cutting one edge, say the one of color 0, yields  $\mathcal{G} \vee e$ , so that we evaluate

$$\begin{aligned} \mathfrak{T}_{\mathcal{G} \vee e}(x, \{\beta_e\}, z, s, q, t) &= \beta_1 \beta_2 \beta_3 x z_1^3 z_2 z_3^3 s w^3 q^3 t^2 \\ &+ (\beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3) x z_1 z_3^4 s w^4 q^6 t^4 \\ &+ (\beta_1 + \beta_2 + \beta_3) x z_3^5 s w^5 q^9 t^6 + z_3^8 s^2 w^8 q^{12} t^8 \end{aligned} \quad (\text{A.2})$$

and contracting the same edge gives

$$\begin{aligned} \mathfrak{T}_{\mathcal{G}/e}(x, \{\beta_e\}, z, s, q, t) &= \beta_1 \beta_2 \beta_3 z_1^6 z_2^4 \\ &+ (\beta_1 \beta_2 + \beta_2 \beta_3 + \beta_1 \beta_3) z_1^3 z_2 z_3^3 s w^3 q^3 t^2 \\ &+ (\beta_1 + \beta_2 + \beta_3) z_1 z_3^4 s w^4 q^6 t^4 + z_3^5 s w^5 q^9 t^6. \end{aligned} \quad (\text{A.3})$$

Thus, we get

$$\mathfrak{T}_{\mathcal{G}} = \mathfrak{T}_{\mathcal{G} \vee e} + x \beta_0 \mathfrak{T}_{\mathcal{G}/e}. \quad (\text{A.4})$$

It should be also emphasized that the contraction of edge is performed in the sense of Definition 27. This definition allows us to improve the active/passive contraction scheme as used in [12].

**Example 2: A planar  $w$ -colored graph.** We can compute  $\mathfrak{T}$  for other types of graphs which are not colored tensor graphs. In a specific instance, consider the graph  $\mathcal{G}$  of Figure 33. It combines both one colored vertex and another type vertex. For simplicity, we change the variables  $(x-1) \rightarrow x$  and  $(y-1) \rightarrow y$ .

By the spanning c-subgraph summation, we get

$$\mathfrak{T}_{\mathcal{G}}(x, y, z, s, q, t) = [x z^{10} s w^5 q^9 t^6 + x y z^9 s w^4 q^6 t^4 + 2 z^2 w^2 q^6 t^4 + 3 y z w q^3 t^2 + y^2] z^{14} s w^5 q^9 t^6. \quad (\text{A.5})$$

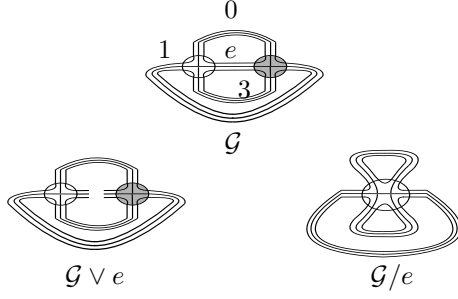


FIGURE 32. A rank 3 colored tensor graph  $\mathcal{G}$  and its cut  $\mathcal{G} \vee e$  and contraction  $\mathcal{G}/e$  for a regular edge  $e$  of color 2.

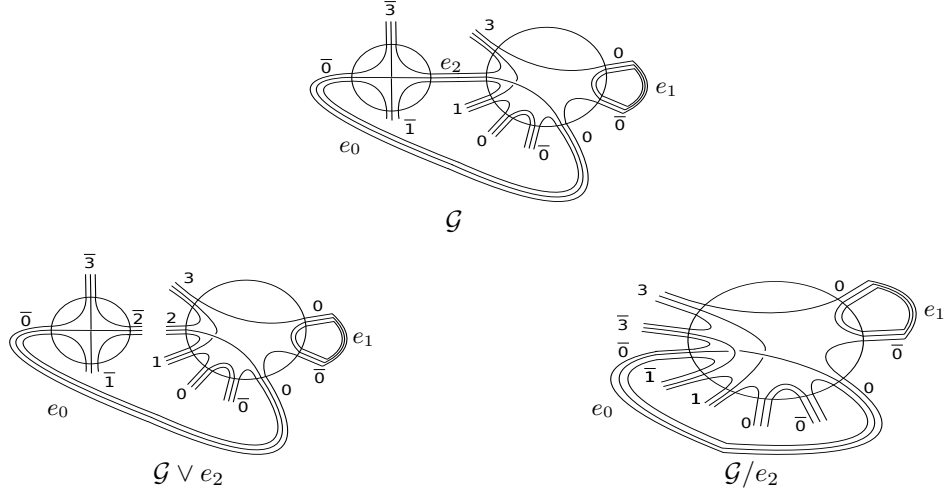


FIGURE 33. A representative graph  $\mathcal{G}$  of a  $w$ -colored graph (displaying conjugate colors at all flags), the cut graph  $\mathcal{G} \vee e_2$  and the contracted graph  $\mathcal{G}/e_2$  with respect to  $e_2$ .

We want to compare (A.5) with the result obtained by contraction and cut procedure using as much as possible results on terminal forms. Using the notations of Figure 33, we must check that

$$\mathfrak{T}_{\mathcal{G}} = \mathfrak{T}_{\mathcal{G} \vee e_2} + \mathfrak{T}_{\mathcal{G}/e_2}. \quad (\text{A.6})$$

Evaluating  $\mathfrak{T}_{\mathcal{G} \vee e_2}$  and  $\mathfrak{T}_{\mathcal{G}/e_2}$ , one finds

$$\mathfrak{T}_{\mathcal{G} \vee e_2} = [xz^8 s(wq)^3 t^2 + 1] \mathfrak{T}_{(\mathcal{G} \vee e_2)/e_0}, \quad \mathfrak{T}_{(\mathcal{G} \vee e_2)/e_0} = \mathfrak{T}_{((\mathcal{G} \vee e_2)/e_0) \vee e_1} + yz \mathfrak{T}_{((\mathcal{G} \vee e_2)/e_0)/e_1}, \quad (\text{A.7})$$

$$\mathfrak{T}_{\mathcal{G}/e_2} = \mathfrak{T}_{(\mathcal{G}/e_2) \vee e_1} + yz \mathfrak{T}_{(\mathcal{G}/e_2)/e_1}. \quad (\text{A.8})$$

We used in (A.7) the fact that  $e_0$  is a bridge in  $\mathcal{G} \vee e_2$  and that  $e_1$  is a 2-inner self-loop in  $(\mathcal{G} \vee e_2)/e_0$ . Meanwhile, in (A.8), we used the fact that  $e_1$  is a 2-inner self-loop in  $\mathcal{G}/e_2$ .

A straightforward calculation yields

$$\begin{aligned} \mathfrak{T}_{((\mathcal{G} \vee e_2)/e_0) \vee e_1} &= z^{16} s w^7 q^{15} t^{10}, & \mathfrak{T}_{((\mathcal{G} \vee e_2)/e_0)/e_1} &= z^{14} s w^6 q^{12} t^8, \\ \mathfrak{T}_{(\mathcal{G}/e_2) \vee e_1} &= z^{16} s w^7 q^{15} t^{10} + yz^{15} s w^6 q^{12} t^8, & \mathfrak{T}_{(\mathcal{G}/e_2)/e_1} &= z^{14} s w^6 q^{12} t^8 + yz^{13} s w^5 q^9 t^6. \end{aligned} \quad (\text{A.9})$$

Plugging these results on (A.7) and (A.8) and summing these contributions in (A.6), we recover (A.5).

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